MHD waves

([3], p.233-239; [1], p.202-237; Chen, Sec.4.18, p.136-144)

We have considered two types of waves in plasma:

1. electromagnetic waves - high-frequency waves affected by motions of electrons in plasma in the oscillating electric field of the waves and in the stationary magnetic field (we assumed that ions are heavy and don't move)

2. electrostatic Langmuir waves and ion-sound waves: oscillations of electrons and ions relative to each other

The third types of waves are MHD waves in which plasma oscillates as a single fluid.

These waves are described by using single-fluid plasma equations and Maxwell equations.

Consider Maxwell equations:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{\text{displacement current}} + \frac{4\pi}{c}\vec{j}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 4\pi q = 0$$

We derive equation for $\vec{E}$ by applying operator $\nabla\times$ to the first equation and substituting $\nabla \times \vec{B}$ from the second equation:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c}\frac{\partial \nabla \times \vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t}$$

Using

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla\cdot)\vec{E} - (\nabla \cdot \nabla)\vec{E} = -\nabla^2 \vec{E}$$

we obtain

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t}$$

where $\vec{j}$ is determined from Ohm's law (lecture 8, p.10-12):

$$\vec{E} + \frac{\vec{v} \times \vec{E}}{c} = \frac{\vec{j}}{\sigma} + \frac{1}{en}\left[\frac{\vec{j} \times \vec{B}}{c} - \nabla p_e\right] + \frac{m_e}{e^2 n}\frac{\partial \vec{j}}{\partial t}$$

In Lecture 7, we considered only slow processes and neglected the last term. However, for high-frequency electromagnetic waves with frequencies close to plasma frequency this term significant.

Plasma velocity $\vec{v}$ is determined from the equation of motion

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{c}(\vec{j} \times \vec{B})$$

, and pressure is determined from the continuity and energy equations:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0$$

$$p/\rho^{\gamma} = \mathrm{const}$$

The last equation is the adiabatic law meaning that the plasma entropy remains constant.

In general, under any given set of circumstances there are four modes at a given frequencies (some of these may be imaginary).

Without magnetic field there are:

A) electromagnetic waves ($\vec{E} \perp \vec{k}$) of two polarization states

B) electrostatic

(a) plasma wave (Langmuir waves) - oscillations of electrons relative to ions

(b) ion-sound waves (electrons and ions move together, inertia is determined by ions, pressure is due to both electrons and ions)

In both cases, the waves propagate if $\omega > \omega_p$. Otherwise, frequency is imaginary, and the waves cannot propagate.

With magnetic field the wave modes are modified, and new type of hydromagnetic waves appear at low frequencies smaller than the ion cyclotron frequency $\omega < \omega_{ci}$. In these waves, inertia is due to ions, and the restoring force is $\vec{j} \times \vec{B}$. These waves can be regarded as waves of the magnetic lines of force, which behave like strings loaded with plasma particles.

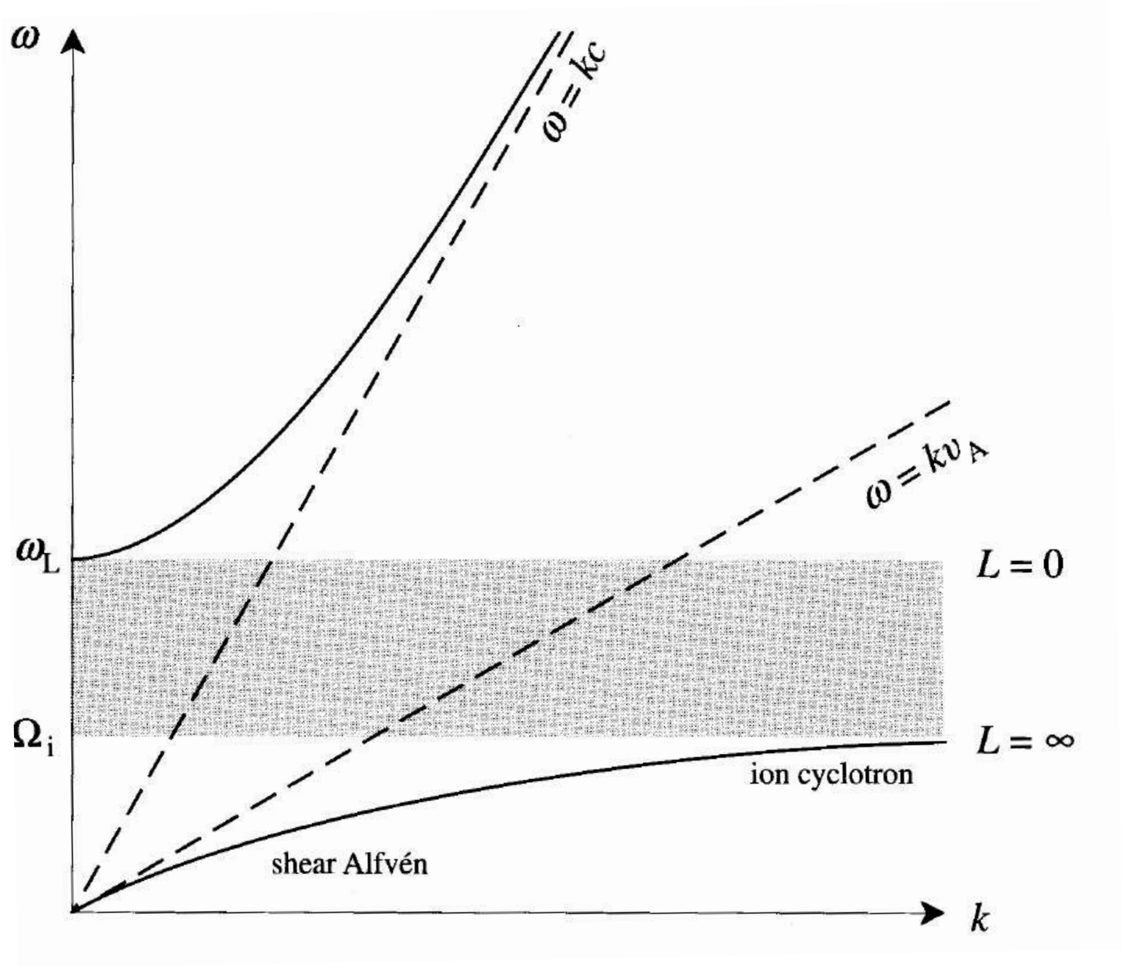

Figure 1: Dispersion diagram for L mode of waves propagating along the magnetic field (including the low-frequency branch of so-called Alfven shear waves). For the left circular polarized waves the polarization vector rotates as in the same direction as the direction of gyration of ions. These waves accelerate ions and cannot propagate with frequencies above the ion cyclotron frequency.

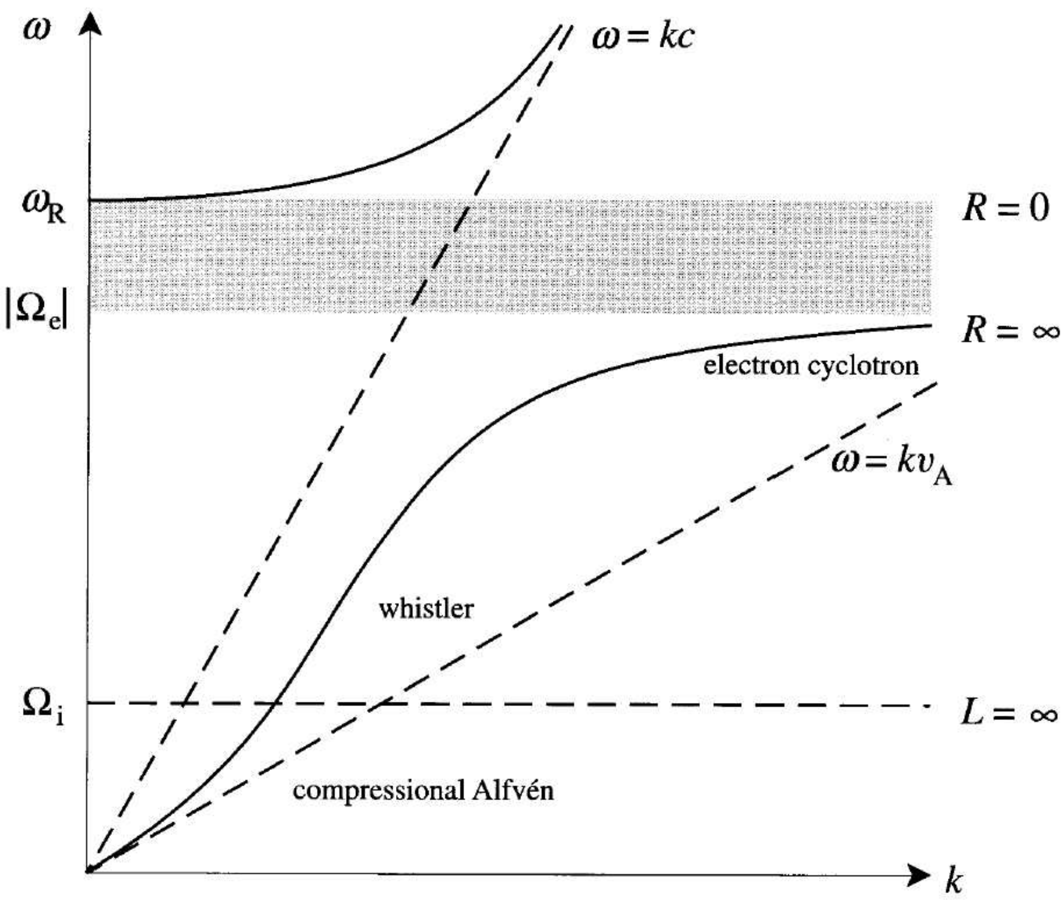

Figure 2: Dispersion diagram for R mode. At low frequencies the whistler wave behave like an Alfven shear wave.

In general, a wave in a magnetic field involves both electric and magnetic forces. A high-frequency wave is a combination of electromagnetic wave with a longitudinal electrostatic wave. Density gradients may produce coupling between different types of waves.

In general, the theory of plasma waves is complicated because of the complexity of the single fluid equations. We consider only simple cases.

For instance, for high-frequency waves without magnetic field we can keep only two terms in the Ohm's law:

$$\vec{E} = \frac{m}{e^2 n} \frac{\partial \vec{j}}{\partial t}$$

Hence the equation for $\vec{E}$ is:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{4\pi e^2 n}{c^2 m} \vec{E}$$

From this obtain the dispersion relation for electromagnetic waves in plasma:

$$\frac{\omega^2}{c^2} = k^2 + \frac{\omega_p^2}{c^2}.$$

Alfven waves

Let us consider now the low-frequency hydromagnetic waves.

In we neglect the displacement current, Hall effect, pressure gradient, and compressibility. Then, the equations have the following form:

$$\nabla^2 \vec{E} = \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t}$$

$$\rho \frac{\partial \vec{v}}{\partial t} = \frac{1}{c}\vec{j} \times \vec{B}$$

If

$$\vec{B} = (0, 0, B_0)$$

$$\vec{v} = (v_x, 0, 0)$$

$$\vec{E} = (0, E_y, 0)$$

$$\vec{j} = (0, j_y, 0)$$

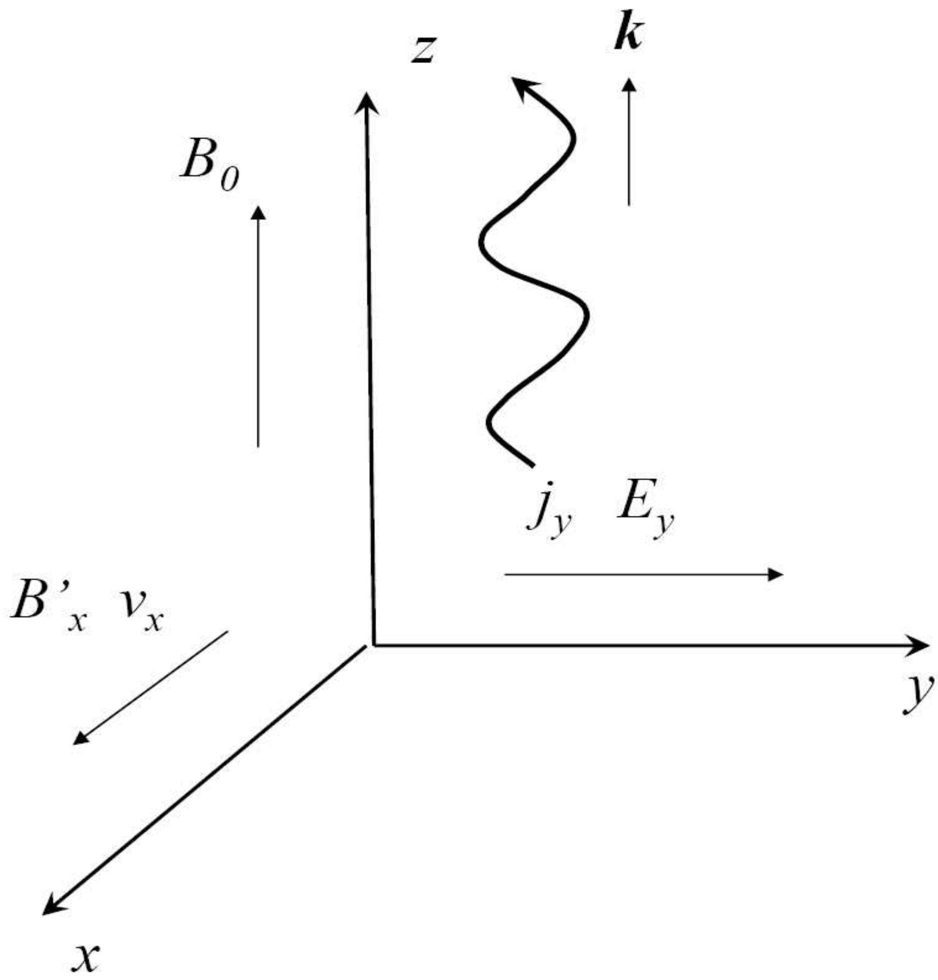

Figure 3: Geometry of an Alfven wave propagating along $\vec{B}_0$.

then

$$\rho \frac{\partial v_x}{\partial t} = \frac{1}{c} j_y B_0$$

$$E_y - \frac{1}{c} v_x B_0 = 0$$

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{4\pi}{c^2} \frac{\partial j_y}{\partial t}$$

$$v_x = \frac{cE_y}{B_0}$$

$$j_y = \frac{c\rho}{B_0}\frac{\partial v_x}{\partial t} = \frac{c^2\rho}{B_0^2}\frac{\partial E_y}{\partial t}$$

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{4\pi\rho}{B_0^2}\frac{\partial^2 E_y}{\partial t^2}$$

Thus, the dispersion relation of these waves (Alfven waves) is

$$\omega^2 = \frac{B_0^2}{4\pi\rho}k^2 = V_a^2 k^2$$

where

$$V_A^2 = \frac{B_0^2}{4\pi\rho}$$

is **the Alfven speed**.

Consider basic properties of Alfven waves. If

$$E_y = E_0 \sin\omega\left(t - \frac{z}{V_A}\right)$$

then

$$j_y = \frac{c^2\rho E_0\omega}{B_0^2}\cos\omega\left(t - \frac{z}{V_A}\right)$$

$$v_x = \frac{cE_0}{B_0}\sin\omega\left(t - \frac{z}{V_A}\right)$$

We find the oscillating magnetic field of the wave from the Maxwell equation

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

the y-component of which has the following form

$$\frac{\partial B_x}{\partial z} = \frac{4\pi}{c} j_y$$

Hence

$$\frac{\partial B_x}{\partial z} = \frac{4\pi c\rho}{B_0^2}\frac{\partial E_y}{\partial t}$$

Substituting E_y and integrating over z we get

$$B_x = -\frac{cE_0}{V_A}\sin\omega\left(t - \frac{z}{V_A}\right)$$

We see that v_x and B_x oscillate in antiphase. We can calculate the kinetic and magnetic energy densities averaged over the wave period, taking into account that:

$$\left\langle \sin^2\omega\left(t - \frac{z}{V_A}\right)\right\rangle = \frac{1}{2}$$

$$\left\langle \frac{\rho v_x^2}{2}\right\rangle = \frac{\rho c^2 E_0^2}{4B_0^2}$$

$$\left\langle \frac{B_x^2}{8\pi} \right\rangle = \frac{c^2 E_0^2 \rho}{4 B_0^2}$$

Thus, the kinetic and magnetic energies of Alfven waves are equal.

For the relative amplitudes of velocity and magnetic field oscillations we obtain

$$\frac{|v_x|}{V_A} = \frac{|B_x|}{B_0} = \frac{cE_0}{B_0}$$

If v_x is large then B_x is also large. Hence Alfven waves can amplify the initial magnetic field and transport it to large distances. However, the condition of incompressibility requires that the Alfven speed is much smaller than the speed of sound $V_A \ll c_s$.

Consider now the equation for the magnetic lines of force:

$$\frac{dx}{B_x} = \frac{dz}{B_0}$$

$$\frac{dx}{dz} = \frac{B_x}{B_0} = -\frac{cE_0}{B_0 V_A} \sin \omega (t - z/V_A)$$

The general solution for the lines of force displacement is:

$$x = x_0 + \frac{cE_0}{B_0\omega}\cos\omega(t - z/V_A)$$

The corresponding velocity of the line of force is:

$$\frac{dx}{dt} = -\frac{cE_0}{B_0}\sin\omega(t - z/V_A) = v_x$$

Hence the magnetic field lines are “frozen” into the plasma.

When the electrical resistivity of plasma is zero ($\sigma = \infty$) the waves are non-dissipative, otherwise the the Alfven waves dissipate.

We can calculate the averaged over the period Joule dissipation and the corresponding change of the wave energy:

$$\frac{dW}{dt} == -\frac{1}{T}\int_0^T \frac{j^2}{\sigma}dt$$

where

$$W = \frac{1}{T}\int_0^T \left(\frac{1}{2}\rho v_x^2 + \frac{B_x^2}{8\pi}\right)dt$$

We obtain

$$W = \frac{\rho c^2 E_0^2}{2B_0^2}$$

$$\left\langle j^2 \right\rangle = \frac{c^4 \rho^2 E_0^2 \omega^2}{B_0^4} = \frac{2W\omega^2 \rho c^2}{B_0^2}$$

$$\frac{dW}{dt} = -\frac{2W\omega^2 \rho c^2}{\sigma B_0^2} = -\frac{W\omega^2 c^2}{2\pi\sigma V_A^2} = -\frac{W}{\tau}$$

where

$$\tau = \frac{2\pi\sigma V_A^2}{c^2 \omega^2} = \frac{V_A^2}{2\omega^2 \nu_m} = \frac{L^2}{\nu_m}$$

Here

$$\nu_m = \frac{4\pi\sigma}{c^2}$$

is called "magnetic viscosity",

$$L = \frac{V_A}{\sqrt{2}\omega} = \frac{\lambda}{2\sqrt{2}\pi}$$

is a "characteristic size" of variations in plasma, $\lambda = 2\pi/k = 2\pi V_A/\omega$ is the wavelength.

The dissipation rate relative to the wave period is

$$\tau/P = \tau\omega/2\pi = \frac{V_A^2}{4\pi\omega\nu_m} = \frac{1}{2\sqrt{2}\pi}\frac{V_A L}{\nu_m} = \frac{1}{2\sqrt{2}\pi} Re_m$$

where

$$Re_m = \frac{V_A L}{\nu_m}$$

is the magnetic Reynolds number. It determines the relative time scale of the Joule dissipation compared to the dynamic time scale.

Magnetic Reynolds number

It plays a fundamental role in the plasma MHD theory. Consider the equation for the magnetic field evolution in the presence of Joule dissipation

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

$$\vec{j} = \sigma\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E} = \frac{\vec{j}}{\sigma} - \frac{(\vec{v} \times \vec{B})}{c}$$

$$\vec{j} = \frac{c}{4\pi}\nabla \times \vec{B}$$

Finally, we obtain

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \nabla \times \left[\frac{c^2}{4\pi\sigma}(\nabla \times \vec{B})\right]$$

The magnetic Reynolds number determines the relative role of the two terms in the right-hand side: magnetic field advection and dissipation.

The relative importance of these terms for a process of a characteristic scale L, velocity v is determined by the magnetic Reynolds number:

$$R_M = \frac{\frac{vB}{L}}{\frac{c^2}{4\pi\sigma}\frac{B}{L^2}} = \frac{4\pi\sigma L v}{c^2}.$$

For typical coronal conditions: $T = 10^6$K, $\sigma = 10^{12}$ s^{-1}, $L = 10^8$ cm, $v = 10^5$ cm/s,

$$R_M \sim 10^4 >> 1.$$

For uniform σ the last term can be simplified:

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}.$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma}\nabla^2 \vec{B}$$

Then, if $\vec{v} = 0$ we get a diffusion equation:

$$\frac{\partial \vec{B}}{\partial t} = D\nabla^2 \vec{B},$$

where

$$D = \frac{c^2}{4\pi\sigma}$$

is a diffusion coefficient for magnetic field.

Exercises:

1. Estimate the characteristic scale of dissipation of magnetic field in solar flares. The duration of solar flares is 10^3 sec.

   $$L \sim \sqrt{\frac{c^2 t}{4\pi\sigma}} \sim 10^5 \text{cm} = 1\text{km}.$$

   This is smaller than the observed flare structure. What does that mean?

2. Estimate the decay time of sunspots ($L \sim 10^9$ cm, $T \sim 10^4$K, $\sigma \sim 10^9$ s^{-1}).

   $$t \sim \frac{4\pi\sigma L^2}{c^2} \sim 10^7 \ \text{sec} \sim 4 \text{ months}.$$

   This is longer the observed lifetime of sunspots. Why?

Frozen Magnetic Flux Approximation

Consider a high-conductivity plasma, $R_M >> 1$, or $\sigma = \infty$ ('ideal plasma'). Then the equations for magnetic field are:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}),$$

$$\nabla \cdot \vec{B} = 0.$$

Consider a 1D case: $\vec{B} = (0, B, 0)$, $\vec{v} = (v, 0, 0)$, - plasma motion across the magnetic field lines, with B and v depending only on x.

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}(vB).$$

From this and the mass equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v),$$

we get

$$\frac{dB}{dt} + B\frac{\partial v}{\partial x} = 0,$$

$$\frac{d\rho}{dt} + \rho\frac{\partial v}{\partial x} = 0.$$

Then

$$\frac{dB}{dt} - \frac{B}{\rho}\frac{d\rho}{dt} = 0,$$

or

$$\frac{d\left(\frac{B}{\rho}\right)}{dt} = 0,$$

or

$$\frac{B}{\rho} = \text{const.}$$

For a general 3D case:

$$\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\nabla\right)\vec{v}.$$

This equation shows that in ideal plasma magnetic field is coupled with plasma density. It follows the plasma motions.

Consider magnetic flux Φ through a plasma area S restricted by a closed curve Γ:

$$\Phi = \int\int_S \vec{B}\cdot d\vec{s},$$

where $\vec{s}$ is a vector perpendicular to the area. If we change the contour line of this plasma element then the total flux will change due the change of the magnetic field strength as follows from the MHD equations and due to the change of the area of this element:

$$\frac{d\Phi}{dt} = \frac{d\Phi'}{dt} + \frac{d\Phi''}{dt},$$

where

$$\frac{d\Phi'}{dt} = \int\int_S \frac{\partial \vec{B}}{\partial t} d\vec{s}.$$

Now consider a small change of the area $d\vec{s}$ due to plasma motion with velocity $\vec{v}$ during time dt:

$$d\vec{s} = \vec{v}dt \times d\vec{l},$$

where $\vec{l}$ is the change of the length of contour Γ. Then the change of the magnetic flux:

$$d\Phi'' = \vec{B} \cdot d\vec{s} = \vec{B} \cdot \vec{v} \times d\vec{l}dt = -dt(\vec{v} \times \vec{B}) \cdot d\vec{l},$$

where we used a vector-product relation. Then,

$$\frac{d\Phi''}{dt} = -\int_\Gamma \vec{v} \times \vec{B} d\vec{l}.$$

Using Stokes' theorem to replace the contour integral with the surface integral we obtain for the total flux:

$$\frac{d\Phi}{dt} = \int\int_S \frac{\partial \vec{B}}{\partial t} d\vec{s} - \int_\Gamma \vec{v} \times \vec{B} d\vec{l},$$

$$\frac{d\Phi}{dt} = \int\int_S \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \right] d\vec{s} = 0.$$

The right-hand side is equal zero because it satisfies the equation for magnetic field in an ideal plasma.

Thus, $\frac{d\Phi}{dt} = 0$. This is the frozen flux theorem: the total magnetic flux through a plasma element does not change under deformations of this element, that is magnetic field moves with the plasma.

Magnetic forces

The MHD momentum equation is

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{c} \vec{j} \times \vec{B} = -\nabla p + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B}$$

or using the vector identity

$$(\nabla \times \vec{B}) \times \vec{B} = (\vec{B}\nabla)\vec{B} - \frac{1}{2} \nabla B^2$$

we obtain

$$\rho\frac{d\vec{v}}{dt} = -\nabla(p + \frac{B^2}{8\pi}) + \frac{1}{4\pi}(\vec{B}\nabla)\vec{B}$$

Hence, magnetic field contributes to plasma pressure and also provides tension force (which tends to make the lines of force straight).

Magnetoacoustic waves

Now we consider a general case of MHD waves including compressibility (variations of ρ) and assuming that the unperturbed plasma is stationary.

From the entropy conservation equation

$$p/\rho^\gamma = \text{const}$$

for small perturbations p' and ρ' we obtain

$$p' = c_s^2\rho'$$

where

$$c_s^2 = \gamma p/\rho$$

is the squared sound speed.

Similarly, from the momentum, induction and continuity equations we have

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -c_s^2 \nabla \rho' + \frac{1}{4\pi} (\nabla \times \vec{B}') \times \vec{B}_0$$

$$\frac{\partial \vec{B}'}{\partial t} = \nabla \times (\vec{v} \times \vec{B}_0)$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \vec{v} = 0$$

We consider plane wave and choose the coordinate system with axis z in the direction of the wave propagation so that:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0, ; \quad \frac{\partial}{\partial z} = ik, \quad \frac{\partial}{\partial t} = -i\omega$$

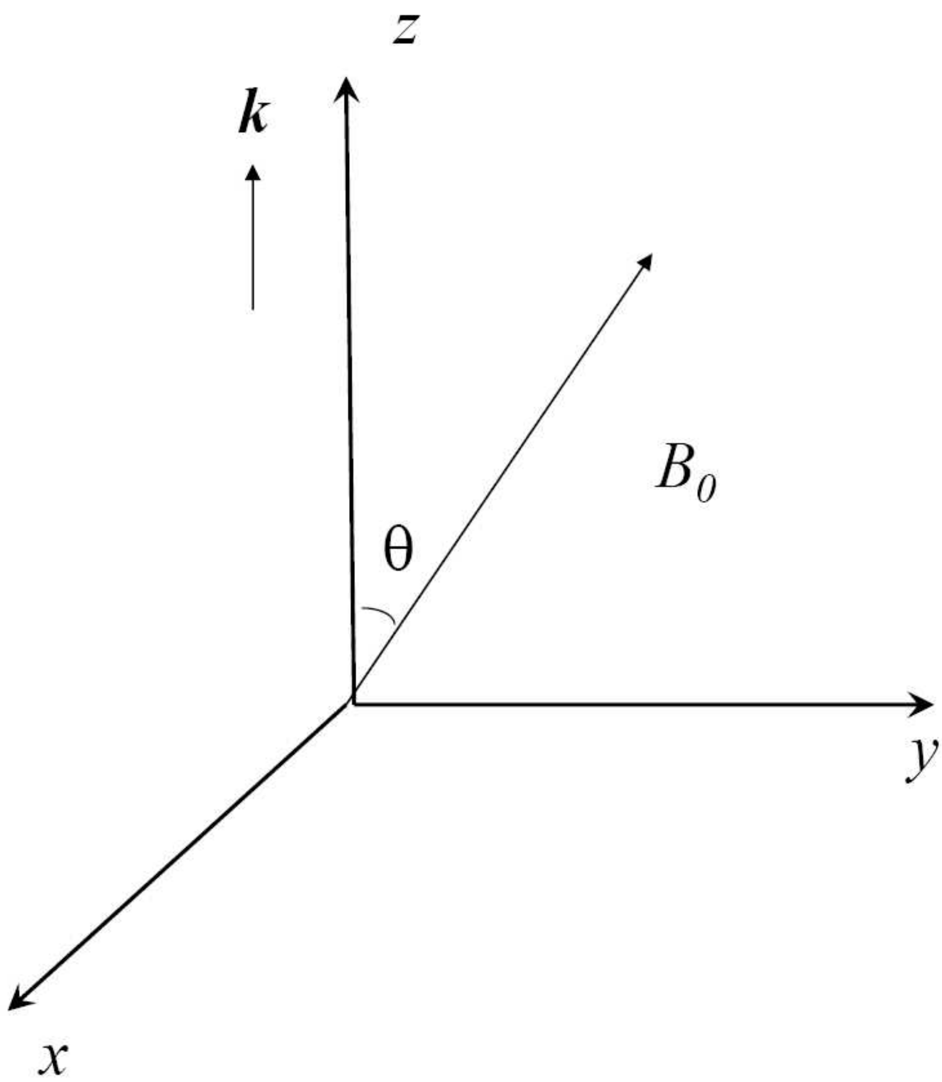

Figure 4: Geometry for magnetoacoustic waves propagating along z-axis.

We obtain the following system of equations

$$\omega\rho_0 v_x + \frac{B_0}{4\pi} k B_x \cos\theta = 0$$

$$\omega\rho_0 v_y + \frac{B_0}{4\pi} k B_y \cos\theta = 0$$

$$\omega\rho_0 v_z - k c_s^2 \rho' - \frac{B_0}{4\pi} k B_y \sin\theta = 0$$

$$\omega B_x + B_0 k v_x \cos\theta = 0$$

$$\omega B_y + B_0 k v_y \cos\theta - B_0 k v_z \sin\theta = 0$$

$$\omega B_z = 0$$

$$\omega\rho' - k\rho_0 v_z = 0$$

$B_z = 0$ means that the magnetic field variations in the wave are perpendicular to the direction of propagation $\vec{k}$.

The first and fourth equations form a separate system from which we get the dispersion relation:

$$\omega^2 \rho_0 - k^2 \frac{B_0^2}{4\pi} \cos^2\theta = 0$$

or

$$\omega/k = V_A \cos\theta$$

This is the Alfven wave propagating in z direction with plasma motions along the x axis and $\rho' = 0$. In this wave

$$\vec{v}' = -\frac{\vec{B}'}{\sqrt{4\pi\rho_0}}.$$

The determinant of the other 4 equations gives:

$$u^4 - (V_A^2 + c_s^2)u^2 + V_A^2 c_s^2 \cos^2\theta = 0$$

where $u = \omega/k$.

This equation has two solutions

$$u_{1,2}^2 = \frac{1}{2}(V_A^2 + c_s^2) \pm \frac{1}{2}\sqrt{V_A^4 + c_s^4 - 2V_A^2 c_s^2 \cos 2\theta}$$

corresponding to fast (+) and slow (-) magnetoacoustic waves.

For $\theta = 0$ (propagation along the magnetic field):

$$u_{1,2}^2 = \frac{1}{2}(V_A^2 + c_s^2) \pm \frac{1}{2}|V_A^2 - c_s^2|$$

If $V_A > c_s$ then $u_1 = V_a$ and $u_2 = c_s$, the fast wave propagates with the Alfven speed, and the slow wave with the sound speed.

For $\theta = \pi/2$ (propagation perpendicular to the magnetic field) we have

$$u_1 = \pm\sqrt{V_A^2 + c_s^2}, \quad u_2 = 0$$

hence, only the fast MHD wave propagates across magnetic field.

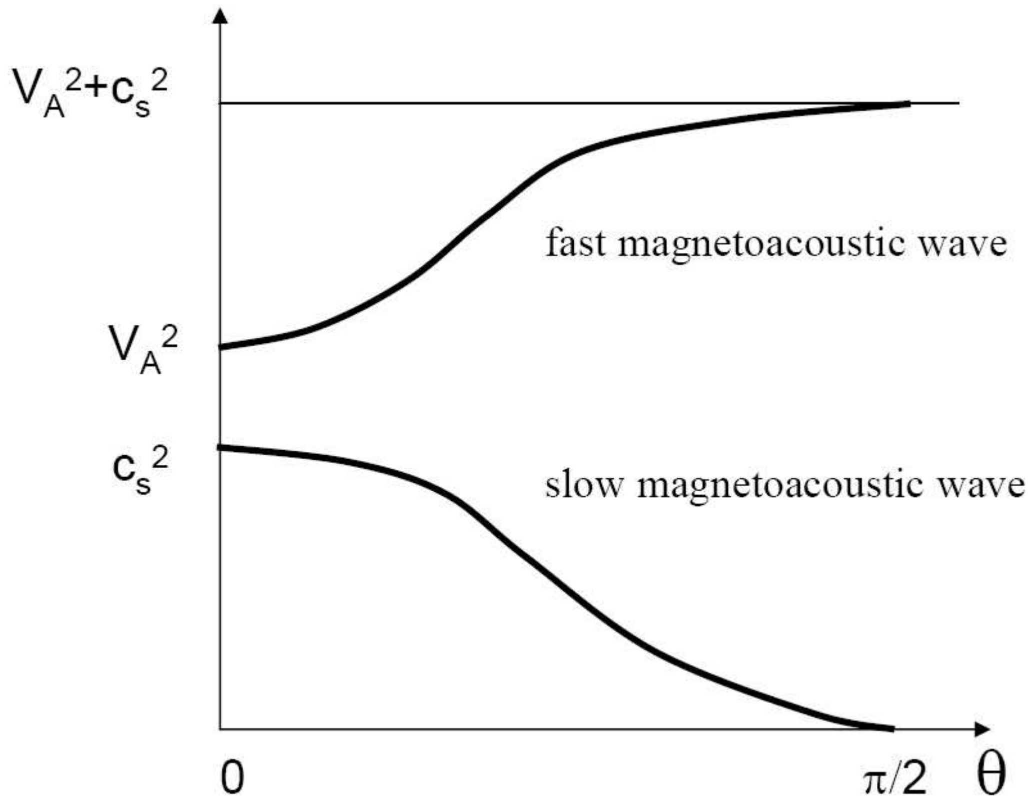

Figure 5: The square of the phase speed of the fast and slow magnetoacoustic waves as a function of angle between the propagation vector and the magnetic field vector for the case $V_A > c_s$.

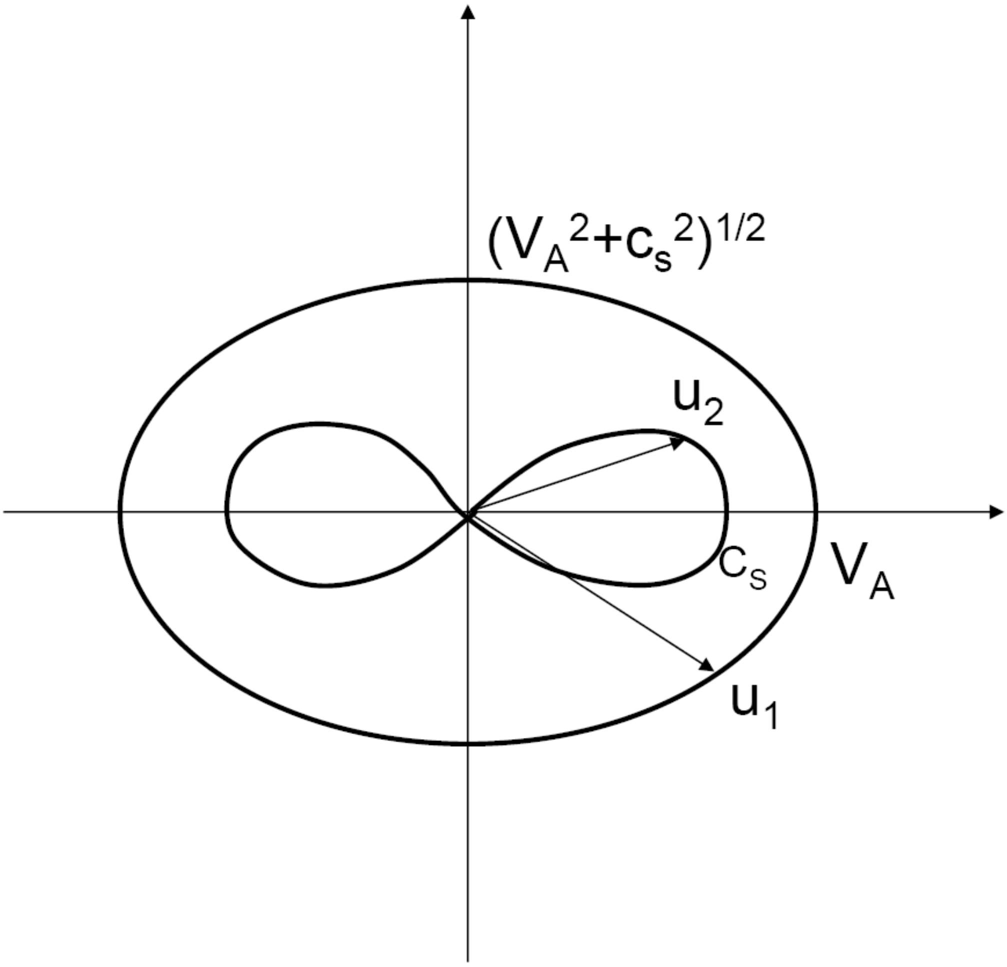

Figure 6: Polar diagram of the phase velocity vector for the fast (u_1) and slow (u_2) waves.

In the magnetoacoustic waves plasma oscillates under pressure and magnetic forces. When they add up we get the fast waves, when they subtract we get the slow waves.

For all the waves there is the equipartition of energy

$$\left\langle \frac{\rho v^2}{2} \right\rangle = \left\langle \frac{B'^2}{8\pi} \right\rangle + \left\langle \frac{p'^2}{2\rho c^2} \right\rangle$$

that is the mean kinetic energy is equal to the sum of magnetic energy and compression energy.