Thermonuclear Fusion. Tokamak.

([1], p.2-6; [4], p.129-135)

Controlled "hydrogen bomb" was suggested in 1952. The main nuclear reactions are with hydrogen isotopes, deuterium D and tritium T:

$$D+T\rightarrow {}^4He+n+17.6\ Mev$$

In this reaction the form of kinetic energy:
3.5 MeV for ^{4}He and 14.1 MeV for n.

$$D+D\rightarrow \begin{cases} {}^3He+n+3.2\ MeV \\ T+n+4.0\ MeV \end{cases}$$

Deuterium (one proton and one neutron) is stable and found in the form of heavy water D_2O. In sea water, there 1 molecule of D_2O for every 6500 molecules of H_2O. The energy that can obtained from 1 gallon of water from the reactions with D is 300 times greater than the energy from 1 gallon of gasoline.

Tritium (one proton + two neutrons) is radioactive, emits β-particles and has half-life of 12.3 years. Therefore, tritium has to be produced, e.g., in

reactions of fast neutrons with Li:

^{7}Li + n$\rightarrow$ ^{4}He+T+n - 2.47 MeV

The new slow neutron can react with ^{6}Li:

^{6}Li+n$\rightarrow$ ^{4}He + T+ 4.8 MeV

As a result the process has positive energy output, and each fast neutron can produce more than one atom T.

However, reactions D+T and D+D produce fast particles and cause radioactivity of the reactor walls. Reaction with ^{3}He

D+^{3}He$\rightarrow$^{4}He+p+15.5 MeV

does not cause much radioactivity. This is the best reactions for nuclear synthesis. There is almost no ^{3}He on Earth, but it is formed on the moon.

For a nuclear reaction two nuclei have to be brought close to each other, so that the nuclear forces begin to work. This distance is about $r_0 \sim 10^{-13}$ cm. For larger distances the Coulomb forces are dominant, and the particle have to have

energy higher than the electrostatic barrier

$$U_0 \approx \frac{e^2}{r_0} \sim 1 \text{ MeV}$$

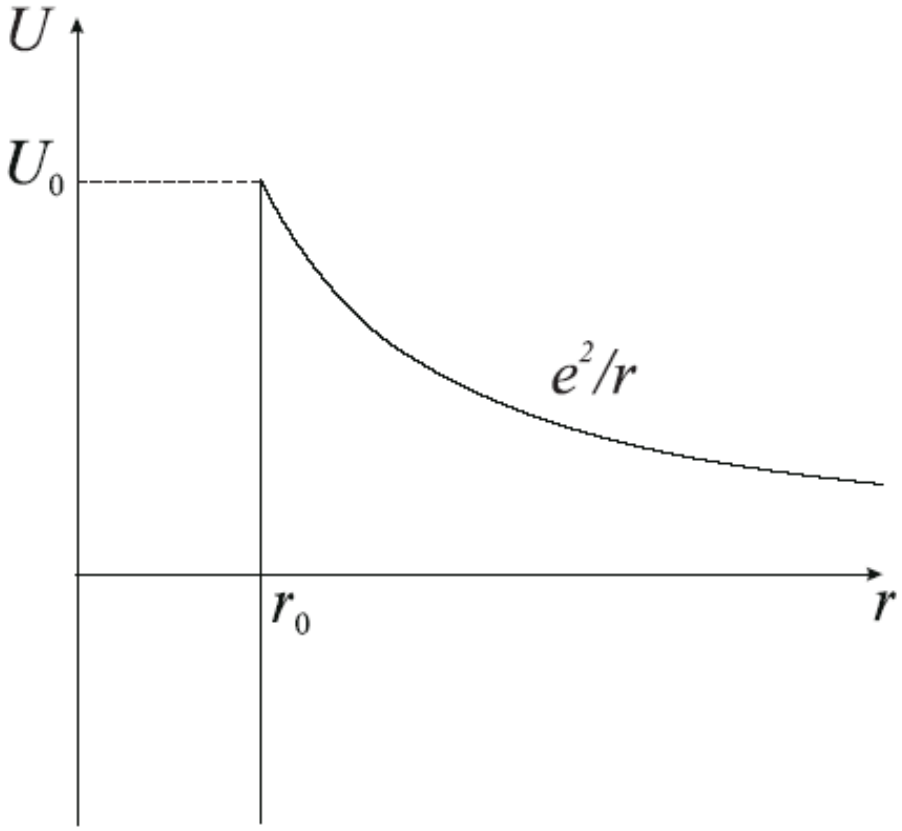


Figure 1: Coulomb potential for collisions of two nuclei.

In stars, where the plasma temperature is about 10^7 K or 1 keV, these reactions occur via the

tunneling effect, and the cross-section is small:

$$\sigma \propto \frac{1}{E} \exp\left(-\frac{\text{const}}{E^{1/3}}\right)$$

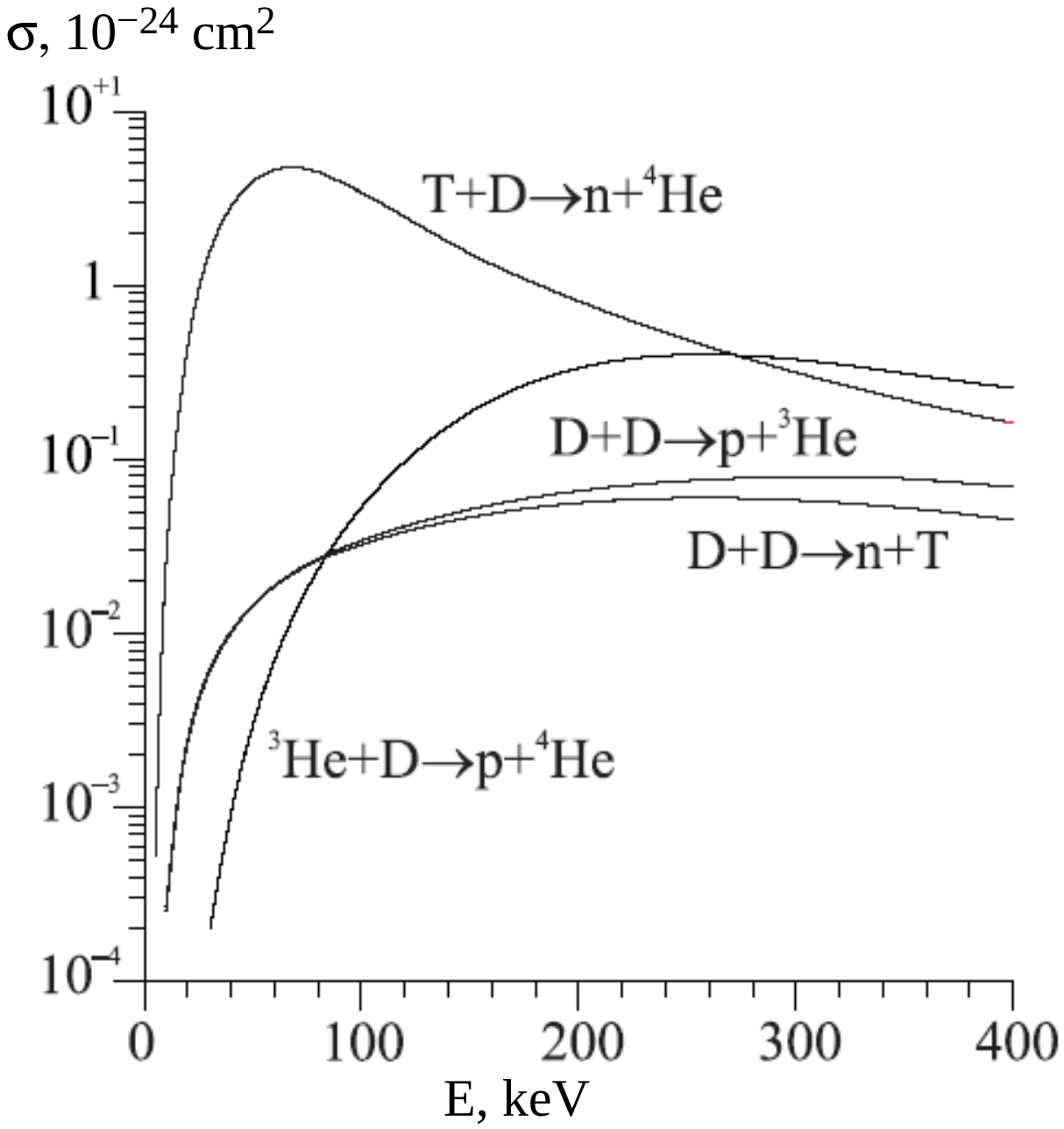


Figure 2: Cross-sections of the fusion reactions.

At low energy $E < 100$ keV the maximum cross-section of D+T reactions (σ_{DT}) is two order of magnitude higher than the cross-section of D+D reaction, and reaches about $5 \cdot 10^{-24}$ cm^2. This is much higher than the cross-section expected from the radius of the nuclear forces $\pi r_0^2 \sim 3 \cdot 10^{-26}$ cm^2. This is because of a resonant character of the D+T reactions, which is occurs through a formation of long-living nuclear cluster.

The number of D+T reactions in unit volume and in unit time for nuclei density

$$n_D = n_T = \frac{n}{2}$$

is estimated as

$$g = \frac{1}{4} n^2 \langle \sigma_{DT} v \rangle$$

where $v = |\vec{v}_T - \vec{v}_D|$ is the relative velocity of the nuclei.

The reaction rate per particle $\langle \sigma_{DT} v \rangle$ is growing function of energy. Thus the main contribution comes from superthermal particles.

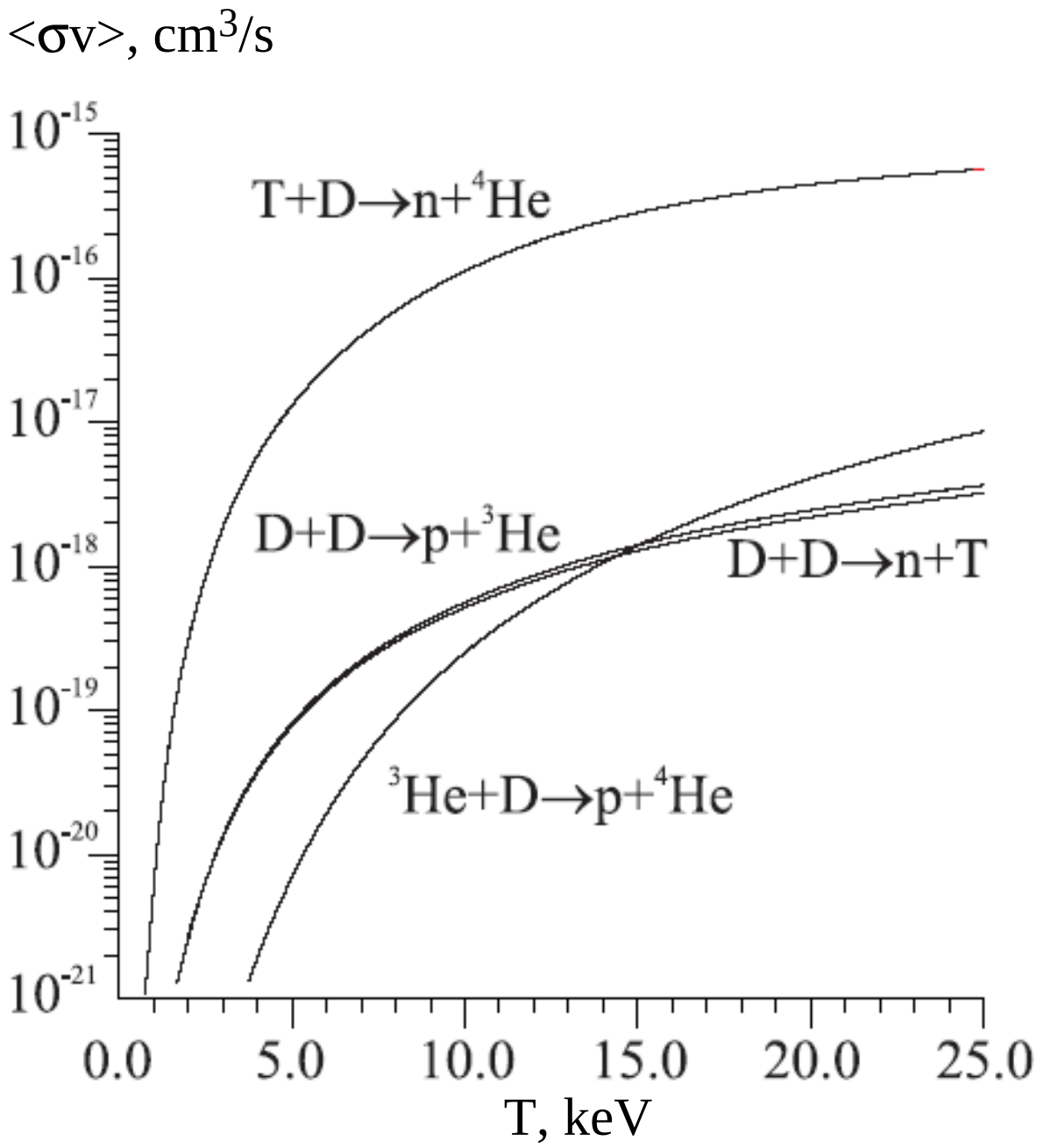


Figure 3: Rates the fusion reactions as a function of plasma temperature.

Consider the condition of the positive energy output in a nuclear reactor, that is the energy

released in nuclear reactions exceeds the energy used to create the plasma.

The plasma energy losses are due bremsstrahlung radiation. Since

$$P_{brem} \propto n^2$$

and

$$P_{nuc} \propto n^2$$

the ratio P_{nuc}/P_{brem} depends only on temperature. The condition

$$P_{nuc}/P_{brem} > 1$$

is satisfied when plasma temperature $T > 3$ keV.

However, in practice, the main source of the energy loss is due to the loss of particles from plasma. If the life-time of a particle in the plasma is τ. Then the energy loss rate is

$$\frac{3nT}{\tau}$$

(because each ions takes $3T/2$ and each electron takes also $3T/2$). Hence, the condition of the

positive energy output is

$$P_{nucl} = \frac{1}{4} n^2 \langle \sigma_{DT} v \rangle Q_{DT} > \frac{3nT}{\tau}$$

where $Q_{DT} = 17.6$ MeV. Thus, for the positive energy output in a thermonuclear reactor we have to satisfy

$$n\tau > \frac{12T}{\langle \sigma_{DT} v \rangle Q_{DT}} = f(T)$$

where n is the plasma density, τ is the plasma confinement time.

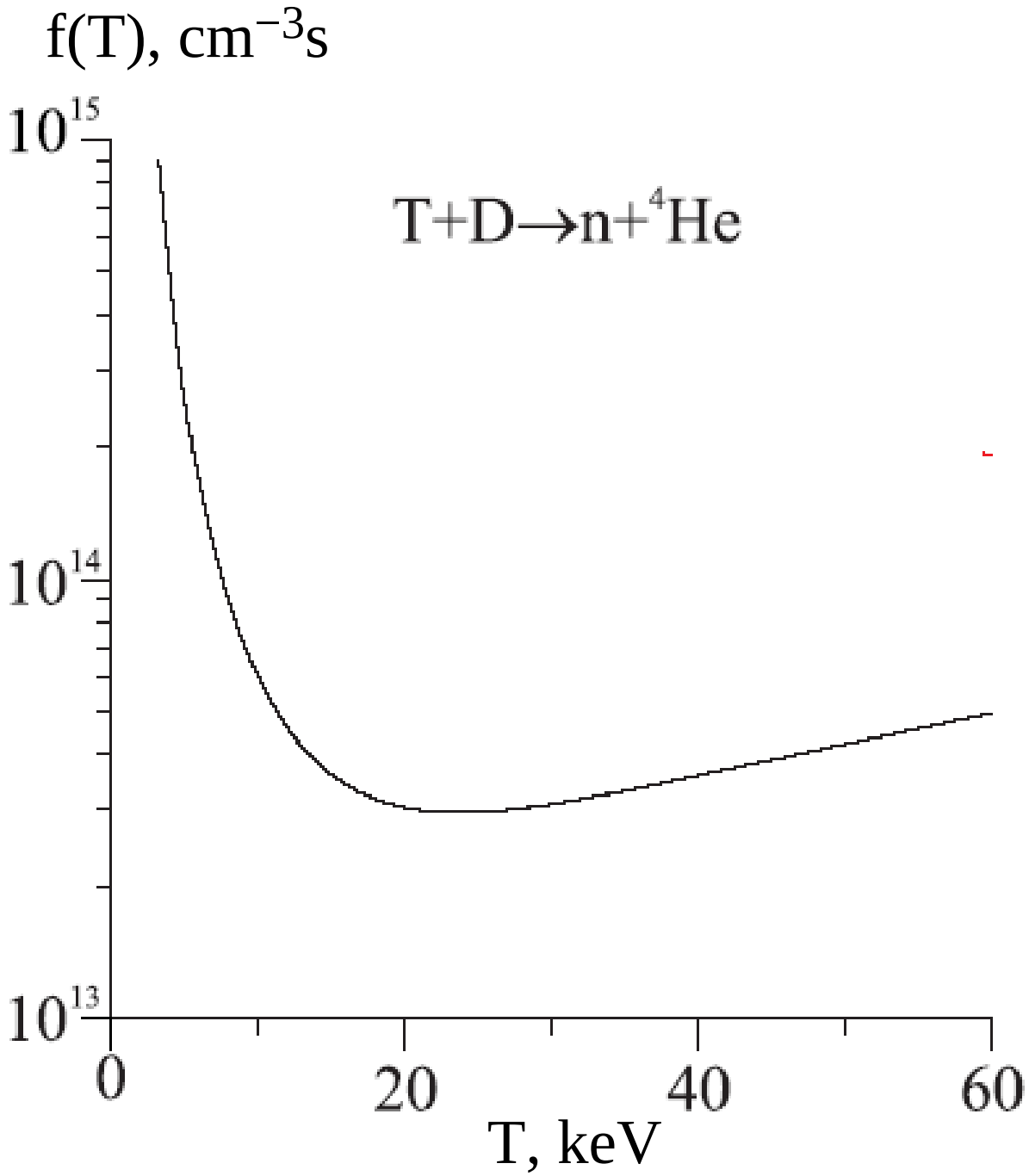


Figure 4: Function $f(T)$.

Function $f(T)$ has a minimum because at low T σ_{DT} is a steep function of T and at high T the

numerator growth as T. The minimum of this function

$$f_{min} = 3 \cdot 10^{13} \ \text{s/cm}^3$$

at $T_{min} = 23.6$ keV$\approx 250 \times 10^6$ K.

In practice it is important to have a lower temperature, e.g. $T = 10$ keV. At this temperature $f(T) \approx 10^{14}$ s/cm^3. Hence, the condition for positive energy output in nuclear fusion is

$$n\tau > 10^{14} \ \text{s/cm}^3$$

This is called **the Lawson criterium**.

There are two basic ways to achieve this:

1. increase τ using magnetic confinement of plasma, e.g. $\tau \sim 1 - 5$ sec,
   $n > 2 \cdot 10^{13} - 10^{14}$ cm^{-3}.

2. increase n using laser compression - inertial fusion, e.g. $\tau \sim 10^{-11}$ sec, $n > 10^{25}$ cm^{-3}.

Magnetic Equilibrium. Tokamaks

A steady solution of plasma with magnetic field is given by the momentum equation with $\vec{v} = 0$:

$$\nabla p = \frac{\vec{j} \times \vec{B}}{c}$$

Since

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

we obtain the equation of magnetostatic equilibrium

$$\nabla p = \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} = \frac{1}{4\pi} \left[(\vec{B} \cdot \nabla) \vec{B} - \nabla \frac{B^2}{2} \right]$$

or equivalently

$$\nabla \left(p + \frac{B^2}{8\pi} \right) = \frac{(\vec{B} \cdot \nabla) \vec{B}}{4\pi}$$

This is 'pressure-balance condition'. The left-hand side describes magnetic pressure perpendicular to magnetic field lines, and the right-hand side describes 'bending and parallel compression'.

The ration of gas pressure to plasma pressure is

called 'plasma β'

$$\beta = \frac{8\pi p}{B^2}$$

In astrophysical plasma $\beta \gg 1$, and in laboratory plasma β is a few percent.

Consider a cylindrical plasma with electric current along the axis, so-called 'cylindrical pinch'.

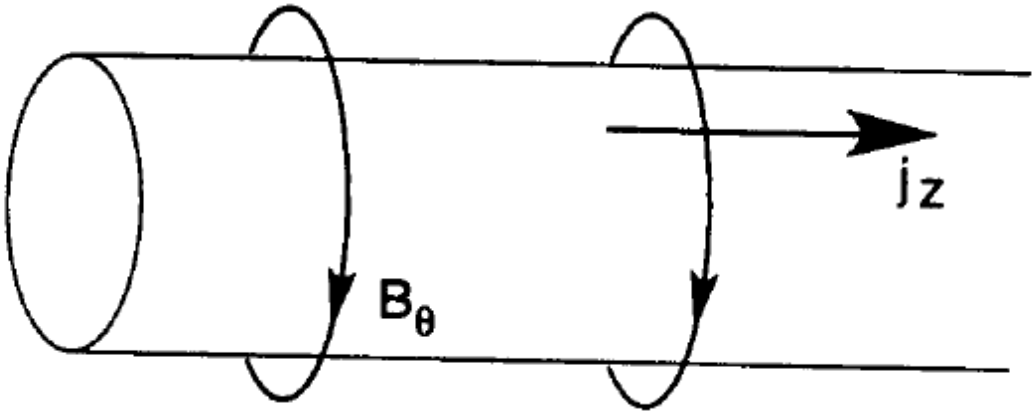


Figure 5: Cylindrical plasma pinch equilibrium in which the azimuthal magnetic field is created by the axial current j_z.

In this case the equilibrium condition in the cylindrical coordinates is

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}.$$

Integrating from 0 to r we obtain

$$p(r) = p_0 - \frac{B_\theta^2}{8\pi} - \frac{1}{4\pi}\int_0^r \frac{B_\theta^2}{r} dr$$

where $p_0 = p(r = 0)$. Consider a plasma of radius a carrying uniformly distributed current surrounded by vacuum

$$j_z(r) = \begin{cases} j_{z0}, & r < a \\ 0, & r > a \end{cases}$$

The total current is

$$I = \pi a^2 j_{z0}$$

From the Ampere's law $\nabla \times \vec{B} = 4\pi\vec{j}/c$ we find the azimuthal magnetic field increases linearly with radius inside the plasma cylinder

$$B_\theta = B_{\theta a}\frac{r}{a}$$

where $B_{\theta a}$ is the magnetic field strength at the plasma boundary. Since the magnetic field is weaker inside the plasma than at the boundary we can say that plasma is 'diamagnetic'.

Inside the plasma

$$\frac{\partial B_\theta}{\partial r} = \text{const}$$

Then, from the equilibrium condition we obtain

$$p(r) = p_0 - \frac{B_\theta^2 r^2}{4\pi a^2}$$

From the Ampere's law

$$j_z = \frac{c}{4\pi}\frac{\partial B_\theta}{\partial r} = \frac{c}{4\pi}\frac{B_{\theta a}}{a}$$

or using the formula for the total current we obtain

$$B_{\theta a} = \frac{4}{ac} I$$

Since at the plasma boundary $p(a) = 0$ we get a relation between the central plasma pressure, total current and plasma pinch radius:

$$p_0 = \frac{4}{\pi c^2 a^2} I^2$$

This is so-called 'pinch condition' describing a magnetically self-confined plasma. However, this equilibrium is strongly unstable, particularly with respect to a 'kink mode'.

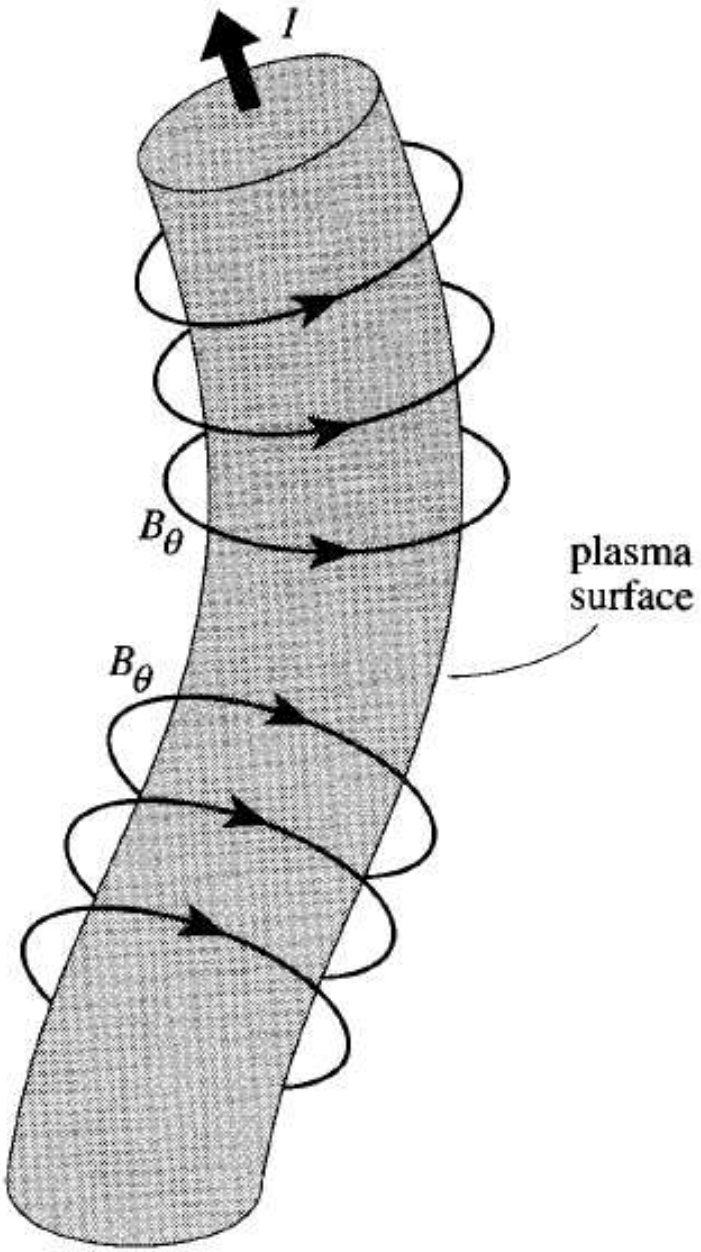


Figure 6: The kink instability. The distortion grows because the magnetic pressure on the concave side of the kink is increased, and on the convex side the magnetic pressure is decreased.

Consider magnetic equilibrium in low-β plasma when the gas pressure is negligible

$$\vec{j} \times \vec{B} = 0$$

Such equilibria are called 'force-free'.

This equilibrium is possible if both axial B_z and azimuthal B_θ components are present. In this case, the pressure-balance condition is

$$\frac{\partial}{\partial r}\left(\frac{B_\theta^2}{2} + \frac{B_z^2}{2}\right) = -\frac{B_\theta^2}{r}$$

This nonlinear equation has many solutions. One of the solution is for uniformly distributed current density j_z, so that

$$B_\theta(r) = B_{\theta a} r/a$$

. Integrating the equilibrium equation we obtain

$$B_z(r)^2 = B_{z0}^2 - B_\theta(r)^2 - 2\int_0^r \frac{B_\theta^2}{r} dr = B_{z0}^2 - 2B_\theta(r)^2$$

Here B_{z0} is the longitudinal filed at the center of the plasma cylinder. This solution is valid inside the plasma cylinder. Outside the cylinder there is current. Therefore, the azimuthal magnetic field

decreases with distance as $1/r$, and B_z must remain constant.

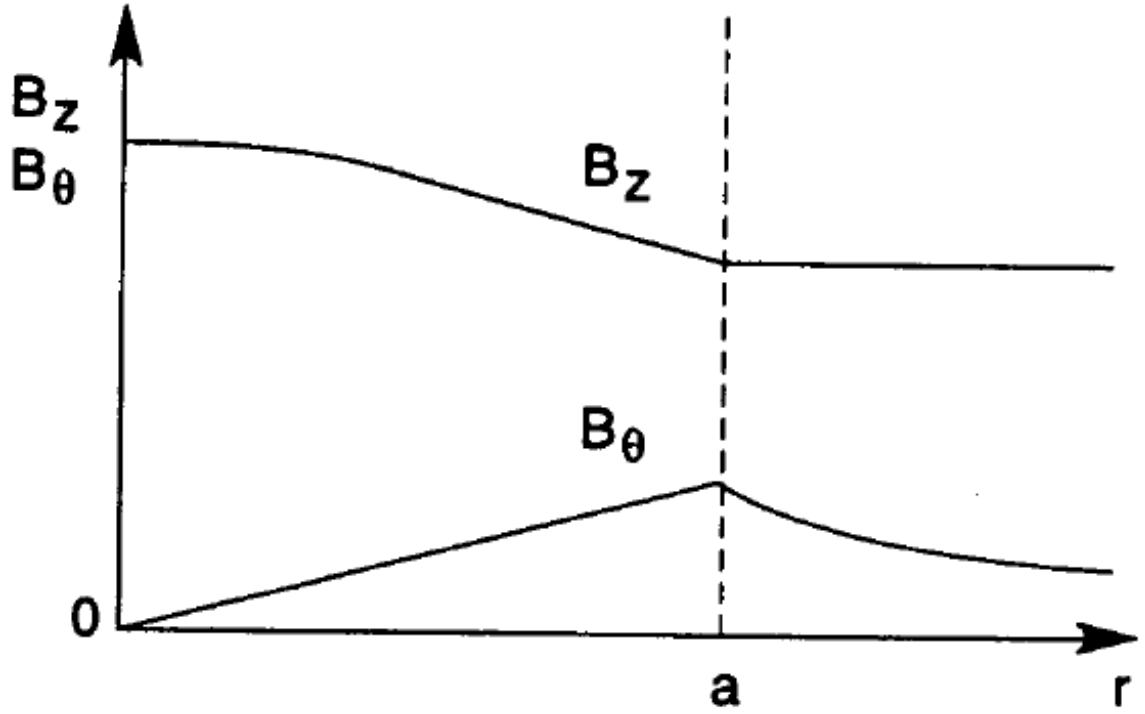


Figure 7: Cylindrical force-free equilibrium. The radius $r = a$ is the edge of the plasma, outside of which there is no electric current, so B_z=const, and $B_\theta \propto 1/r$.

This solution is a cylindrical approximation to 'tokamak' at very low plasma β. The strong field B_z is produced by external coils, and the weaker B_θ field by currents in plasma. In this case magnetic field B_z is stronger inside the plasma

than at the boundaries. Thus plasma is ‘paramagnetic’ in tokamaks.