

Coulomb collisions. Plasma Resistivity.

([11], p.416-421; [2] p.560-588)

The long-range character of Coulomb forces and the mass difference between electrons and ions determine the specific properties of plasma. In atomic or molecular gas particle collisions are characterized by gaskinetic cross sections which are determined by the range a of atomic or molecular forces:

$$\sigma \approx \pi a^2,$$

and typically are of the order of 10^{-15}cm^2 .

In plasma, the Coulomb force, $1/r^2$, does not have a characteristic length scale. Collisions in plasma are understood as interactions between particle which result in substantial change in energy and momentum, with a deflection from their trajectories to a substantial angle of the order of 90° .

Consider collision between two particles with masses of the same order and with charges Z_1e and Z_2e in a plasma of temperature T .

It is clear that a significant transfer energy and momentum will occur when the energy of Coulomb interaction between the particles is comparable with their thermal energy:

$$\frac{Z_1 Z_2 e^2}{r_0} \sim \frac{3}{2} T.$$

This happens at distance:

$$r_0 \sim \frac{2}{3} \frac{Z_1 Z_2 e^2}{T}.$$

The cross section of this collision is:

$$\sigma \approx \pi r_0^2 \approx \frac{4\pi}{9} \frac{Z_1^2 Z_2^2 e^4}{T^2}.$$

However, the problem is more complicated because of the important role of "distant" collisions corresponding to large impact parameter. Distant collisions occur more frequently than the "close" collisions and contribute more to the cross section.

Consider a flux j of particles with charge $Z_1 e$ scattered by a particle with charge $Z_2 e$, and calculate the force acting of the scattering particle.

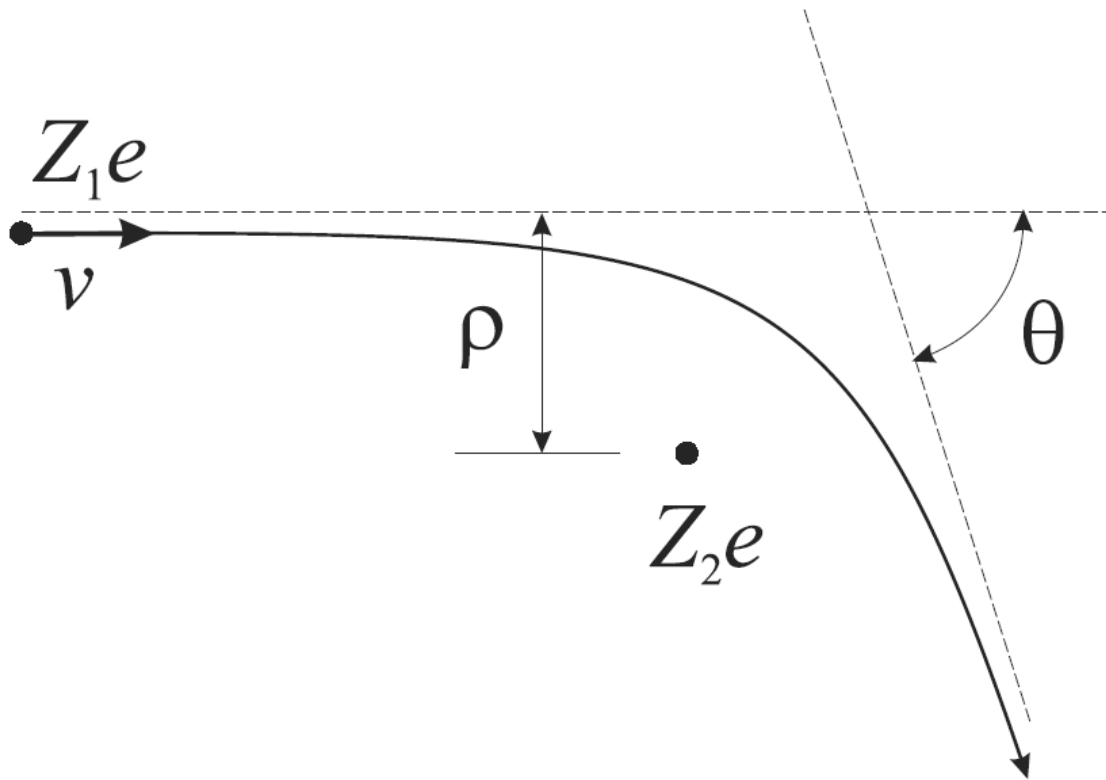


Figure 1: Trajectory of charge Z_1e scattering on a stationary charge Z_2e .

If ρ is the impact parameter (the distance between the original trajectory of Z_1e and charge Z_2e), then the particle flux through a ring of area $2\pi\rho d\rho$ is $2\pi\rho d\rho \cdot j$, where particle flux (the number of particles crossing a unit area in unit time) $j = n_b v$, n_b is the volume density of beam particles.

Each particle changes its trajectory by angle θ

(which depends on ρ) and thus transfer momentum to charge Z_2e :

$$\Delta p_{\parallel} = mv(1 - \cos \theta)$$

in the direction of the particle flux.

Then the total force has the same direction as the initial velocity and is equal to the momentum transferred to Z_2e in unit time:

$$F = \int_0^{\infty} p_{\parallel} 2\pi \rho j d\rho$$

$$F = \int_0^{\infty} mvj(1 - \cos \theta) \cdot 2\pi \rho d\rho = mvj \cdot \sigma_{\text{tr}}$$

where

$$\sigma_{\text{tr}} = 2\pi \int_0^{\infty} (1 - \cos \theta) \rho d\rho$$

is so-called **momentum transport cross section**.

Since $F \sim mv^2 n_b \sigma_{\text{tr}}$, it corresponds to the cross section of an object that absorbs all momentum of incoming particles.

To calculate the cross section we have to determine the scatter angle vs the impact parameter $\theta(\rho)$.

This is easy to calculate for "distant" collisions

(large impact parameter). In this case, $\theta \ll 0$ (weak scattering). Particles keep moving almost along the straight lines. We can calculate the change of the perpendicular component of momentum by integrating the perpendicular component of the Coulomb force:

$$\Delta p_{\perp} = mv\theta = \int_{-\infty}^{\infty} F_{\perp} dt$$

$$F_{\perp} = \frac{Z_1 Z_2 e^2 \rho}{(v^2 t^2 + \rho^2)^{3/2}}$$

$$\Delta p_{\perp} = \int_{-\infty}^{\infty} \frac{Z_1 Z_2 e^2 \rho}{(v^2 t^2 + \rho^2)^{3/2}} dt$$

$$\Delta p_{\perp} = \frac{Z_1 Z_2 e^2}{\rho v} \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^{3/2}} = \frac{2Z_1 Z_2 e^2}{\rho v}$$

From these equations we find:

$$\theta = \frac{2Z_1 Z_2 e^2}{m\rho v^2}$$

The precise Rutherford formula for any ρ is:

$$\tan \frac{\theta}{2} = \frac{Z_1 Z_2 e^2}{m\rho v^2}$$

Calculate the transfer cross section assuming that

most contribution comes from distant impacts (we will check that this is the case). For large ρ or small θ :

$$1 - \cos \theta \approx \frac{1}{2}\theta^2$$

then the momentum transfer cross section is:

$$\begin{aligned} \sigma_{\text{tr}} &= 2\pi \int_0^\infty \frac{\theta^2}{2} \rho d\rho = \frac{4\pi Z_1^2 Z_2^2 e^4}{m^2 v^2} \int_0^\infty \frac{d\rho}{\rho} = \\ &= \frac{4\pi Z_1^2 Z_2^2 e^4}{m^2 v^2} \ln \rho \Big|_0^\infty. \end{aligned}$$

We have singularities at both limits.

Obviously, our approximation is not valid for low distances. The singularity at $\rho \rightarrow 0$ would not occur if we used the precise Rutherford formula.

We can correct the singularity at zero by taking for the lower limit the impact parameter at which the scattering angle is of the order of $\pi/2$. That is

$$\rho_{\text{min}} \simeq \frac{2Z_1 Z_2 e^2}{mv^2}$$

Note that this is the minimal distance at which particles can approach each other ($E_k \sim E_e$). The precise value is not important because this is a logarithmic value.

The singularity at large distances is more fundamental. It cannot be corrected by doing more precise calculations. The reason is that the Coulomb potential decreases with distance too slow. However, we know that the real potential in plasma is not Coulomb potential because of the screening effect. The effective screening distance is given by the Debye radius, r_D . Therefore, we use the Debye radius as the upper limit in this integral:

$$\rho_{\max} \sim r_D = \left(\frac{T}{4\pi n Z_2^2 e^2} \right)^{1/2}$$

Again, the precise value is not important because of log.

Thus, we write the momentum transfer cross section as:

$$\sigma_{\text{tr}} = \frac{4\pi Z_1^2 Z_2^2 e^4}{m^2 v^4} \ln \frac{\rho_{\max}}{\rho_{\min}}$$

$$\Lambda_C \equiv \ln \frac{\rho_{\max}}{\rho_{\min}}$$

is called "Coulomb logarithm".

$$\Lambda_C = \ln \frac{r_D m v^2}{2 Z_1 Z_2 e^2} = \ln \frac{3 T^{3/2}}{2 (4\pi)^{1/2} Z_1 Z_2^2 e^3 n^{1/2}}.$$

Now we can make a simple estimate for $Z_1 = Z_2 = 1$:

$$\Lambda_C \sim \ln \frac{T}{e^2 n} r_D n \sim \ln(r_D^3 n) \sim \ln N_D$$

where N_D is the number of particles in the Debye sphere (plasma parameter). As we know N_D is usually a big number, but $\Lambda_C \approx 10 - 20$ for most cases.

The final formula for the momentum transfer cross section is

$$\sigma_{\text{tr}} = \frac{4\pi Z_1^2 Z_2^2 e^4}{m^2 v^4} \Lambda_C$$

or practical formula:

$$\sigma_{\text{tr}} \simeq \frac{10^{-12}}{E_{\text{eV}}^2} \text{ cm}^2.$$

Knowing the cross section we can calculate the rate of the momentum exchange.

First, we consider an electron beam colliding with cold plasma. We assume that ions are heavy and don't move. Consider first collisions with ions. If n is the ion density then the force acting on the ions in unit volume is (compare with equation for F in

p.4):

$$mvj\sigma_{\text{tr}} \cdot n$$

The same force acts on the electrons of the beam in the opposite direction for each electron:

$$\vec{F} = -m\vec{v}j\sigma_{\text{tr}}n/n_b$$

where n_b is the electron density of the beam. The particle flux in the beam is: $j = n_b v$. Then,

$$\vec{F} = -n\sigma_{\text{tr}}mv \cdot v \sim -n\sigma_{\text{tr}}mv^2$$

The beam will slow down under this force:

$$\frac{dv}{dt} = \frac{F}{m} = -n\sigma_{\text{tr}}v^2$$

However, because the electron scatter on heavy ions that don't move their energy does not change. The scattering results in increase of angle θ . For small angles the velocity changes in the direction of the beam is

$$\Delta v = v(\cos \theta - 1) \approx -v\theta^2/2$$

Then,

$$-\frac{1}{v} \frac{dv}{dt} \approx \frac{d\theta^2/2}{dt} = n\sigma_{\text{tr}}v$$

This means that the scatter angle increases with time.

The characteristic time of this change due to the momentum loss is

$$\tau = \frac{1}{n\sigma_{\text{tr}}v}$$

The inverse quantity, $1/\tau$, is called the collision frequency.

$$\nu_{ei} = n\sigma_{\text{tr}}v$$

(Note that the force can be estimated as $F = -mv\nu_{ei}$).

The product $v\tau$ is a characteristic travel distance for electrons before they lose most of their momentum. Therefore,

$$\lambda = \frac{1}{n\sigma_{\text{tr}}}$$

is called the mean free path.

Practical formulas:

$$\sigma_{\text{tr}} \simeq \frac{3 \cdot 10^{-13}}{T_{\text{eV}}^2} \text{ cm}^2$$

$$\nu_{ei} = 3 \cdot 10^{-5} \frac{n_{\text{cm}^{-3}}}{T_{\text{eV}}^{3/2}} \text{ s}^{-1}.$$

$$\lambda = \frac{3 \cdot 10^{12} T_{\text{eV}}^2}{n_{\text{cm}^{-3}}} \text{ cm}.$$

Consider now collisions of the electron beam with plasma electrons, or more general case for particles with masses m_1 and m_2 and charges Z_1 and Z_2 .

In this case, we have to solve equation for particles type 1 scattering on a particle 2:

$$m_1 \ddot{\vec{r}}_1 = Z_1 Z_2 e^2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3},$$

where \vec{r}_1 and \vec{r}_2 are coordinates of the particles.

To solve this we consider motions in the coordinates of the center of mass, the position of which is:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

If $\vec{r} = \vec{r}_1 - \vec{r}_2$ then

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r}_2$$

The center of mass moves with a constant velocity because there is no external force, $\ddot{\vec{R}} = 0$.

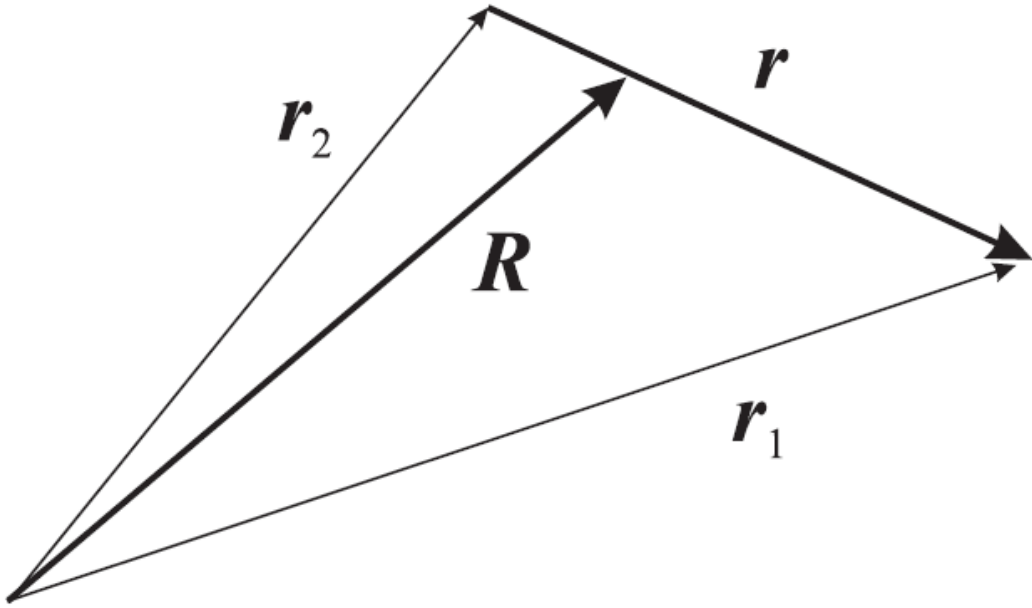


Figure 2: Coordinates r_1 and r_2 of particles 1 and 2, and coordinate R of the reduced particle.

Then, equation for \vec{r}_1 is;

$$m_1 \ddot{\vec{r}}_1 = \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = \frac{Z_1 Z_2 e^2 \vec{r}}{r^3}$$

or

$$m_1 \ddot{\vec{r}}_1 = m_{12} \ddot{\vec{r}} = \frac{Z_1 Z_2 e^2 \vec{r}}{r^3}$$

where $m_{12} = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

Thus the force acting on a reduced particle is equal to the force acting on particle 1. If \vec{v} is velocity of

particle 1 at infinity then

$$\vec{F} = m_{12} \vec{v} \cdot v \sigma_{\text{tr}} n = \frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C}{m_{12} v^3} n \vec{v}$$

For e-e collisions:

$$m_1 = m_2 = m_e, \quad m_{12} = m_e/2.$$

Therefore, the force of electron beam acting is of the same order of magnitude for plasma electrons and ions.

Now we calculate the energy transfer in e-e collisions. In the laboratory coordinate system the mass center moves with constant velocity

$$\vec{v}_{cen} = \dot{\vec{R}} = \frac{m_1 \vec{v}}{m_1 + m_2}$$

where \vec{v} is the velocity of the particles relative to each other.

$$\vec{v}_1 = \frac{m_2 \vec{v}}{m_1 + m_2}$$

The reduced particle does not change its velocity, and thus does not lose energy. There is no energy exchange in the center-of-mass coordinate system because velocity of particle 1 in this system is

constant

$$\vec{v}_1 = \frac{m_2 \vec{v}}{m_1 + m_2}$$

However, in the laboratory system, the work done by the force acting on the reduced particle is:

$$Fv_c = Fv \frac{m_1}{m_1 + m_2}$$

This takes energy from the beam particles, 1, and transfers it to plasma electrons, 2.

The energy of beam particles change with the rate

$$W = -Fv_c$$

$$W = -\frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C n}{m_2 v}$$

or

$$W = -\frac{m_1}{m_2} m_1 v^2 \cdot nv \frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C}{m_1^2 v^4}$$

$$W = -\frac{m_1}{m_2} m_1 v^2 \cdot (nv\sigma_{tr}) = -\frac{m_1}{m_2} m_1 v^2 \cdot \nu_{12}$$

That means that the fraction energy transfer at each $1 \rightarrow 2$ collision is proportional to the mass ratio. This means that the energy transfer between electrons is much faster than between electrons and ions.

So far we considered cool plasma. What happens if plasma temperature is not zero? In this case we consider averages over the velocity distribution of plasma particles.

Consider the case when $v_T \gg v_{\text{beam}}$.

Consider a group of plasma particles with velocity \vec{v}' . Their number is

$$dn = n f(\vec{v}') d^3 v'$$

where $f(\vec{v}')$ is the distribution function; $f(\vec{v}') d^3 v'$ is the probability of plasma particles to have velocity around \vec{v}' .

Then the drag force for beam particles by this group of plasma particles is (cf p.13):

$$d\vec{F} = (\text{Force per 1 particle}) \cdot dn$$

$$d\vec{F} = - \frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C n}{m_{12}} \frac{\vec{v} - \vec{v}'}{|\vec{v} - \vec{v}'|^3} f(\vec{v}') d^3 v'$$

Then we integrate over the velocities \vec{v}' of plasma particles:

$$\vec{F} = \int d\vec{F} = - \frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C}{m_{12}} \vec{I}$$

where

$$\vec{I} = \int \frac{\vec{v} - \vec{v}'}{|\vec{v} - \vec{v}'|^3} f(\vec{v}') d^3 v'$$

For an isotropic distribution function

$$\vec{I} = \frac{\vec{v}}{v^3} \int_0^v f(v') 4\pi v'^2 dv'$$

Since we assumed that the beam velocity is much less than the particle velocity in plasma (thermal velocity) then

$$\vec{I} \simeq \frac{\vec{v}}{v^3} \frac{4\pi v^3}{3} f(0)$$

For Maxwellian distribution function:

$$f\vec{v}' = \left(\frac{m_2}{2\pi T}\right)^{3/2} \exp\left(-\frac{m_2 v'^2}{2T}\right)$$

$$\vec{I} = \vec{v} \frac{4\pi}{3} \left(\frac{m_2}{2\pi T}\right)^{3/2} = \frac{\sqrt{2}}{3} \frac{\vec{v}}{\sqrt{\pi}} \left(\frac{m_2}{T}\right)^{3/2}$$

Then, the drag force in the case of "hot" plasma ($v \ll v_T$) is

$$\vec{F} = -\frac{4\sqrt{2}}{3} \frac{Z_1^2 Z_2^2 e^4 \Lambda_C}{m_{12}} \left(\frac{m_2}{T}\right)^{3/2} \vec{v}$$

Plasma relaxation.

Suppose that the distribution is not Maxwellian. Then the plasma relaxes to thermal equilibrium.

The fastest is the relaxation time for electrons:

$$\tau_1 \sim \tau_{ee}$$

where τ_{ee} is electron-electron collision time.

Then, the ion plasma component relaxes to equilibrium:

$$\tau_2 \sim \tau_{ii} \sim \sqrt{\frac{M}{m}} \tau_{ee}$$

(because $\tau_{ii} = 1/n\sigma_{tr}v \sim \sqrt{M}$ since $v \sim 1/\sqrt{M}$).

The slowest process is e-i relaxation:

$$\tau_{ei} \sim \frac{M}{m} \tau_{ee}.$$

Electric current

Consider motion of electron in electric field:

$$e\vec{E} + \vec{F} = 0$$

where F is the drag force.

Suppose u is a characteristic velocity of electrons then

$$F \sim m u \nu_{ei}$$

$$eE = m u \nu_{ei}$$

$$u = \frac{eE}{m \nu_{ei}}$$

$$j = ne u = \frac{ne^2}{m \nu_{ei}} E = \frac{\omega_p^2}{4\pi \nu_{ei}} E$$

$$\nu_{ei} \sim n \sigma_{tr} v_{Te}$$

If $u < v_T$ then

$$\sigma_{tr} \sim e^4 \Lambda_C / T_e^2.$$

$$F \sim u$$

The drag force increases with the electron velocity.

If $u > v_{T_e}$ then

$$F \sim u^{-2}$$

The drag force decreases with velocity.

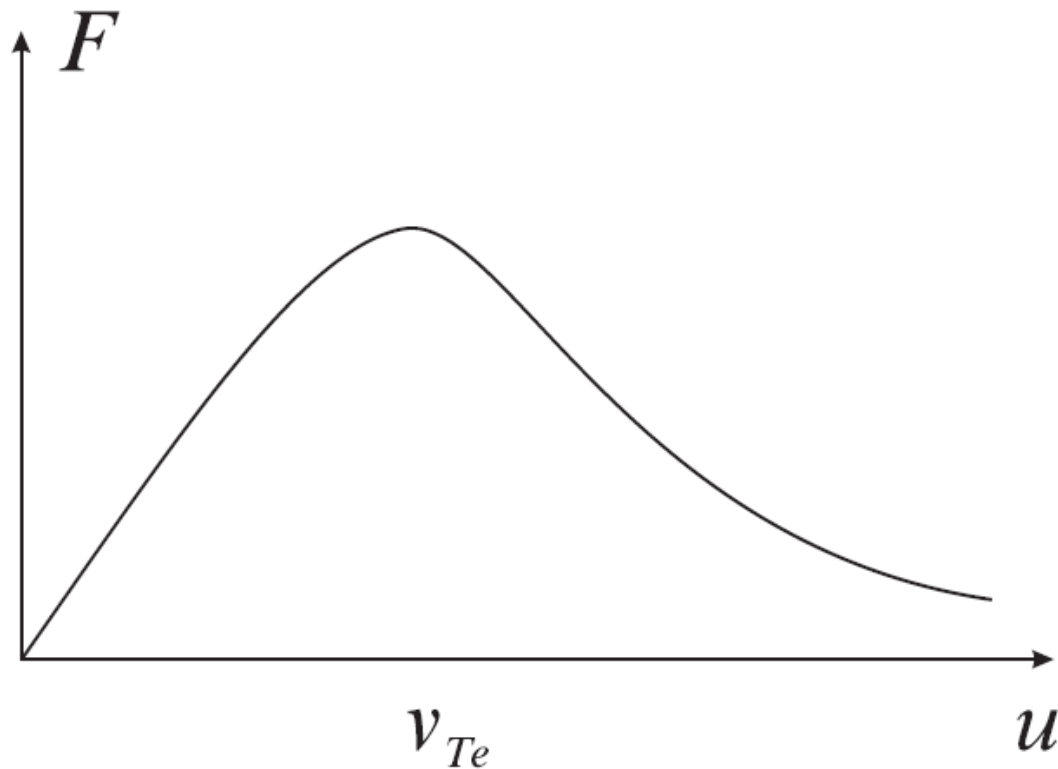


Figure 3: The drag force as a function of velocity u .

If $E > E_{Dr}$ then there is no balance between the acceleration in electric field and the drag force.

This leads to run-away electrons.

$$E_{Dr} \sim \frac{ne^3 \Lambda_C}{T_e} \sim \Lambda_C \frac{e}{r_D^2}$$

E_{Dr} is called Dreicer field.