Particle motion in magnetic field.

([3], P.19-31, [4], P.21-35)

The equation of motion of charge q in static uniform electric and magnetic fields is:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{1}{c}[\vec{v}\times\vec{B}]\right)$$

The change of particle energy:

$$\vec{F}\cdot\vec{v} = q\cdot\vec{v}\cdot\vec{E} + \frac{q}{c}\vec{v}\cdot[\vec{v}\times\vec{B}] = q\cdot\vec{v}\cdot\vec{E}$$

because $\vec{v}\cdot[\vec{v}\times\vec{B}] = 0$. Magnetic field does not change the particle energy.

Consider uniform magnetic field $\vec{B} = (0,0,B)$, and $\vec{E} = 0$:

$$m\frac{d\vec{v}}{dt} = q\frac{\vec{v}\times\vec{B}}{c}$$

$$\vec{v}\times\vec{B} = \vec{i}v_yB - \vec{j}v_xB$$

$$\dot{v}_x = \frac{qB}{mc}v_y$$

$$\dot{v}_y = -\frac{qB}{mc}v_x$$

$$\dot{v}_z = 0$$

The last equation means that there is no force along the magnetic vector. Only motion perpendicular to the magnetic field is affected.

From the first two equations:

$$\ddot{v}_x = \frac{qB}{mc}\dot{v}_y = -\left(\frac{qB}{mc}\right)^2 v_x$$

$$\ddot{v}_x + \omega_c^2 v_x = 0$$

where

$$\omega_c = \frac{|q|B}{mc}$$

is the *cyclotron frequency.*

Note that in the relativistic case: $m \to m\gamma$ where $\gamma = 1/\sqrt{1 - v^2/c^2}$, thus

$$\omega_c = \frac{|q|B}{\gamma mc}$$

The general solution for v_x and v_y is:

$$v_x = v_\perp \exp i\omega_c t$$

$$v_y = \frac{mc}{qB}\dot{v}_x = i \cdot \mathrm{sign}(q) v_\perp \exp i\omega_c t$$

The particle displacement (taking the real parts) is:

$$x = x_0 + \frac{v_\perp}{\omega_c} \sin \omega_c t$$

$$y = y_0 + \frac{v_\perp}{\omega_c} \mathrm{sign}(q) \cos(\omega_c t)$$

These equations describe circular motion in the xy-plane with angular frequency ω_c and radius

$$r_L = \frac{v_\perp}{\omega_c} = \frac{mcv_\perp}{|q|B}$$

is so-called *Larmour radius* or gyroradius.

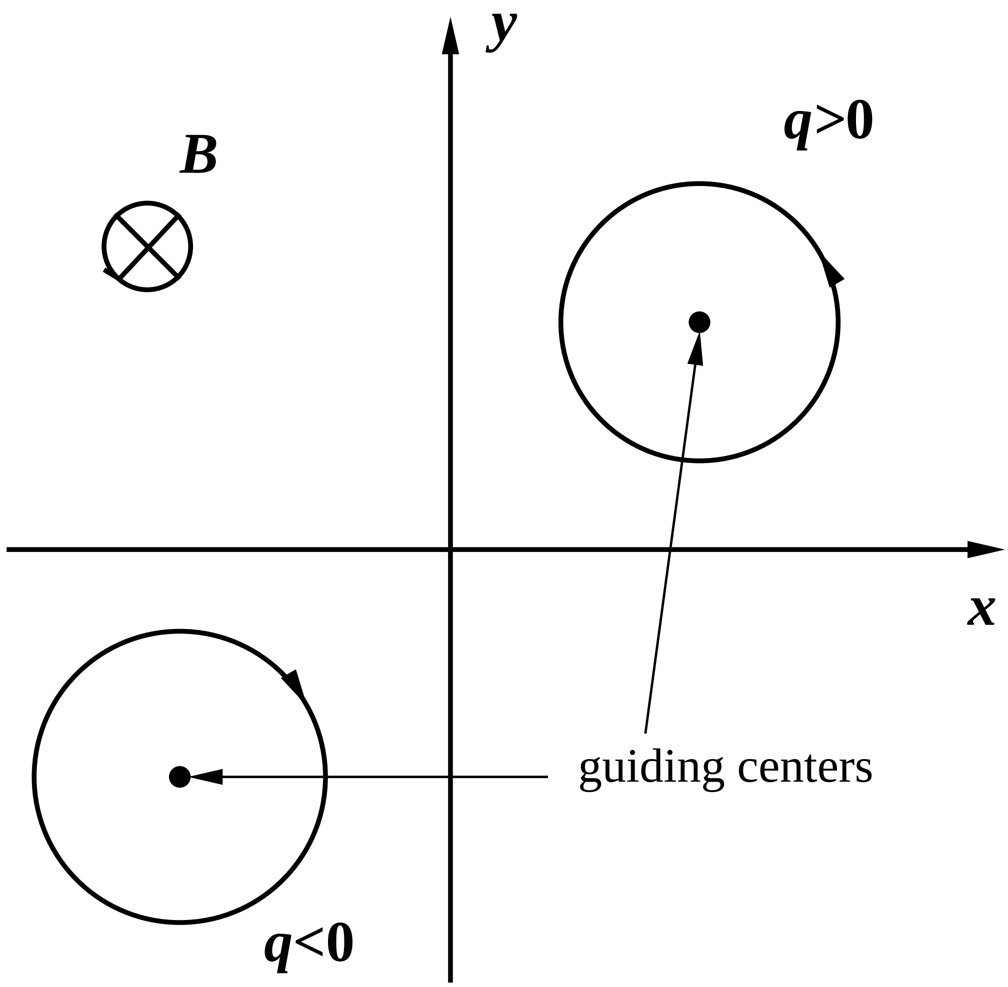

Figure 1:

Magnetic field generated by moving particles in magnetic field is opposite to the original field. This is *diamagnetic property* of plasma. In general, the particle motion is spiral around their *guiding centers*.

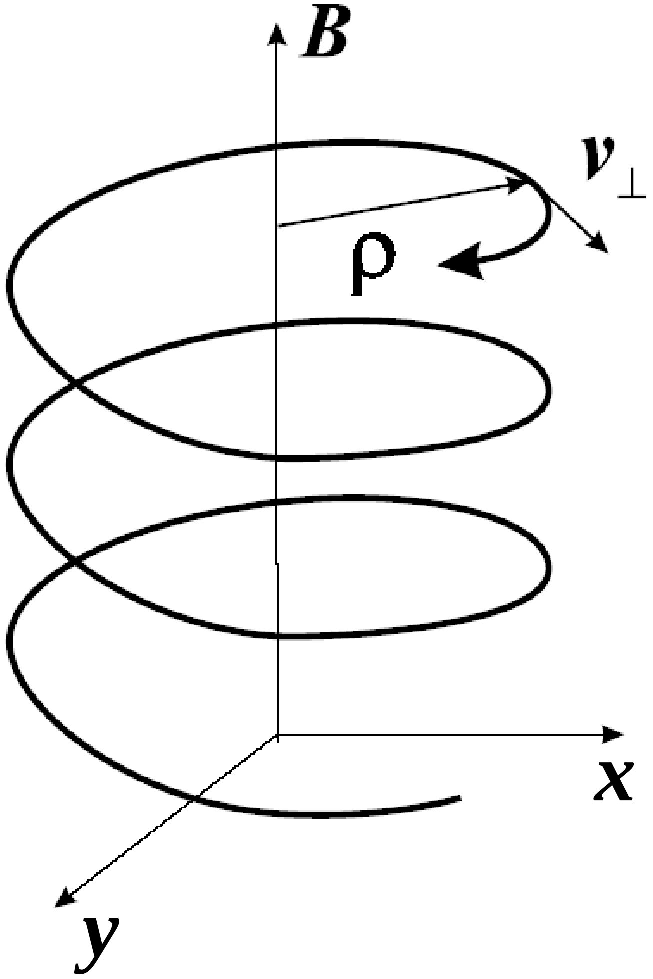

Figure 2: Spiral motion in magnetic field.

Numerical example: Solar corona $B = 10$G, $T = 10^6$K.

$$v_T = \sqrt{3kT/2m} \sim 4 \cdot 10^8 \text{ cm/s}$$

$$\omega_c \simeq 1.6 \cdot 10^8 \text{ rad/s}$$

$$\nu_c = \omega_c/2\pi \sim 2.8 \cdot 10^7 \text{ Hz} = 28\text{MHz}$$

$$r_L \simeq 2.5 \text{ cm}$$

Particle drift in electric and magnetic fields

Consider motion of a charged particle in magnetic and electric fields.

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right)$$

For the velocity components parallel and perpendicular to the magnetic field we get:

$$\frac{dv_{\|}}{dt} = \frac{qE_{\|}}{m}$$

$$\frac{dv_\perp}{dt} = \frac{q}{m}\left[\vec{E}_\perp + \frac{1}{c}\vec{v}_\perp \times \vec{B}\right]$$

Consider the possibility that the transverse velocity can be separated into a constant component ("drift" motion) and a time-varying component ("gyro" motion):

$$\vec{v}_\perp = \vec{v}_d + \vec{v}_g(t)$$

Then

$$\frac{d\vec{v}_g}{dt} = \frac{q}{m}\left(\vec{E}_\perp + \frac{1}{c}\vec{v}_d \times \vec{B} + \frac{1}{c}\vec{v}_g \times \vec{B}\right)$$

The first terms in the right-hand side do not depend on time. Separating out the time-independent and time-dependent components we obtain:

$$\vec{E}_\perp + \frac{1}{c}\vec{v}_d \times \vec{B} = 0$$

and

$$\frac{d\vec{v}_g}{dt} = \frac{q}{m}\frac{1}{c}\vec{v}_g \times \vec{B}$$

Then, make a cross-product of $\vec{B}$ and the first of these two and use

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}),$$

taking into account that $\vec{v}_d \perp \vec{B}$:

$$\vec{B} \times \left(\vec{E}_\perp + \frac{1}{c} \vec{v}_d \times \vec{B} \right) = 0$$

$$\vec{B} \times \vec{E}_\perp + \frac{B^2}{c} \vec{v}_d = 0$$

$$\vec{v}_d = c \left(\frac{\vec{E}_\perp \times \vec{B}}{B^2} \right)$$

This solution is only valid if $\vec{E}_\perp < \vec{B}$. If $\vec{E}_\perp > \vec{B}$ then motion is relativistic, and the electric field is so strong that particle accelerates without gyromotion.

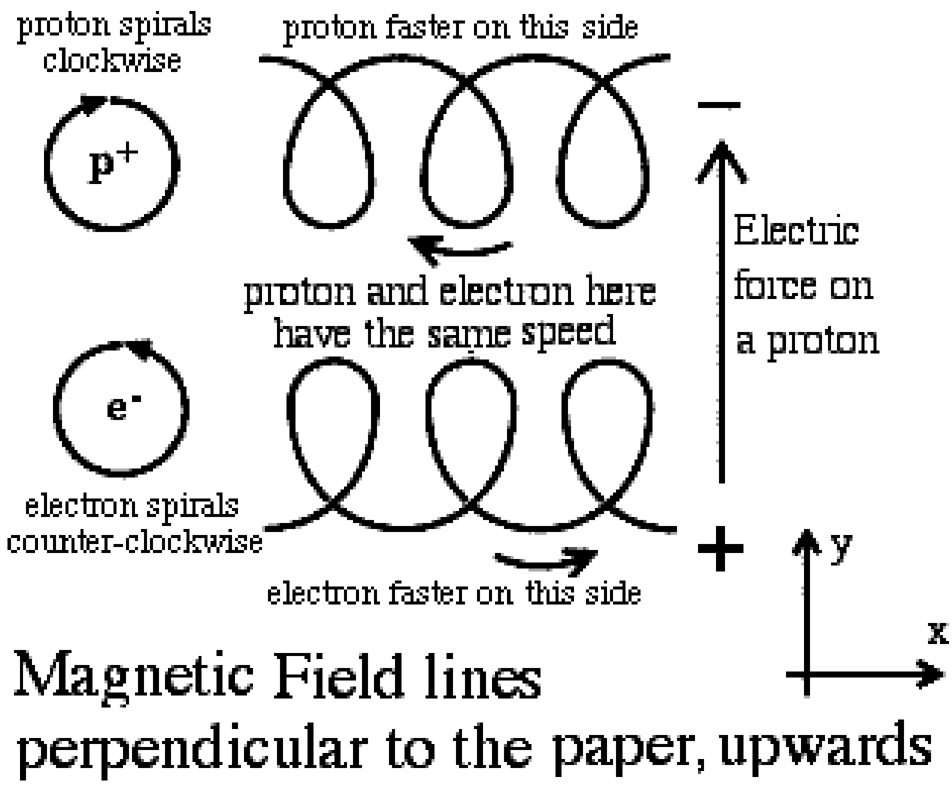

Figure 3: Physical picture of particle drift of crossed electric and magnetic fields.

Electron and protons drift in the same direction. There is no mean electric current in this case.

Particle drift in magnetic and gravitational fields

The equation of motion in magnetic and gravitational fields is:

$$\frac{d\vec{v}}{dt} = \vec{g} + \frac{q}{mc}\vec{v} \times \vec{B}$$

This equation is identical to the equation with an electric field

$$\vec{E}_{\rm eff} = \frac{m}{q}\vec{g}$$

Hence, the drift velocity is:

$$\vec{v}_d = \frac{mc}{q}\frac{\vec{g} \times \vec{B}}{B^2}.$$

In addition to this drift, particles freely accelerate along $\vec{B}$ if $\vec{B} \cdot \vec{g} \neq 0$.

Electron and ions (protons) drift in opposite directions generating electric current:

$$\vec{j} = -ne\vec{v}_{ed} + ne\vec{v}_{pd} = (M + m)cn\frac{\vec{g} \times \vec{B}}{B^2} = \rho c\frac{\vec{g} \times \vec{B}}{B^2}$$

Most current is caused by ions which have higher drift velocity because of the higher mass.

Note that the combined gravity and Lorentz force for the drift current vanishes:

$$\vec{F} = \frac{\vec{j} \times \vec{B}}{c} + \rho\vec{g} = -\rho\vec{g} + \rho\vec{g} = 0.$$

This could be expected because there is no time independent component of acceleration perpendicular to the magnetic field.

Particle drift in non-uniform magnetic field

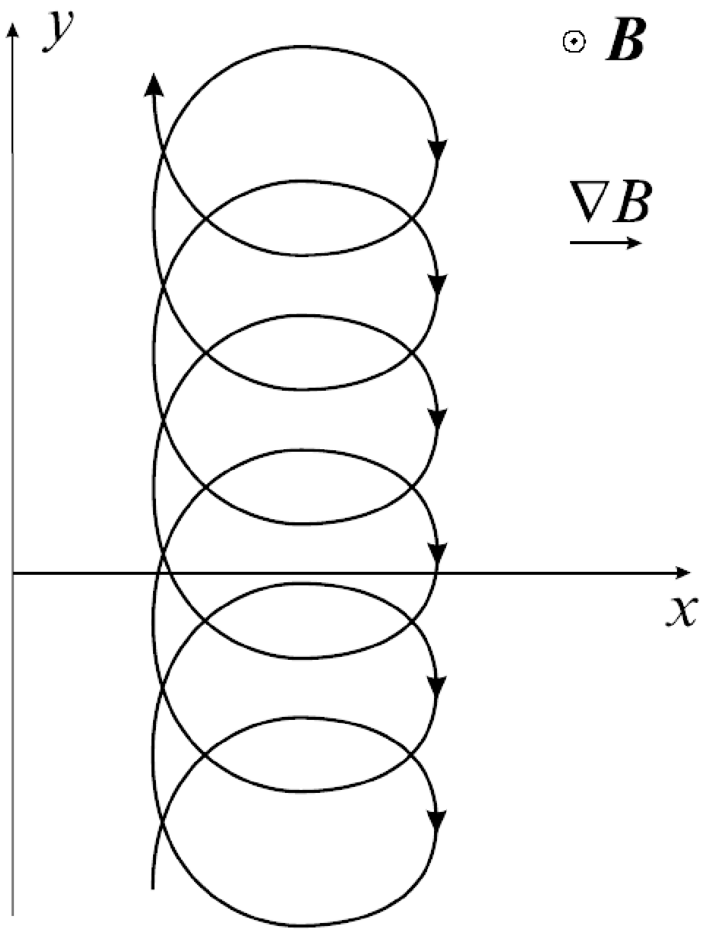

Figure 4: Gradient drift in non-uniform magnetic field.

Because the magnetic is stronger on the right-hand side of the orbit the Larmour radius is smaller than the radius on the left-hand side. This causes displacement of the orbit. The drift velocity should proportional to $v_\perp$, r_L and ∇B.

Consider the Lorentz force:

$$\vec{F} = \frac{q}{c}\vec{v} \times \vec{B}$$

$$F_y = -\frac{q v_x B_z(y)}{c}$$

Because of the gradient:

$$B_z = B_0 + (y - y_0)\frac{\partial B}{\partial y}$$

Hence,

$$F_y = -\frac{q v_x B_0}{c} - \frac{q v_x (y - y_0)}{c}\frac{\partial B}{\partial y}.$$

Now, we average this force over one orbit.

$$v_x = v_\perp \cos\omega_c t$$

$$y - y_0 = \frac{v_\perp}{\omega_c}\mathrm{sign}(q)\cos(\omega_c t)$$

We get: $\overline{v}_x = 0$.

$$v_x(y - y_0) = \frac{v_\perp^2}{\omega_c}\mathrm{sign}(q)\cos^2(\omega_c t)$$

$$\overline{v_x(y-y_0)} = \frac{v_\perp^2}{2\omega_c}\mathrm{sign}(q)$$

The mean force is:

$$\overline{F_y} = -\frac{q}{c}\frac{v_\perp^2}{2\omega_c}\nabla\vec{B}\cdot\mathrm{sign}(q)$$

or

$$\overline{F_y} = -\mathrm{sign}(q)\frac{qv_\perp r_L}{2c}\nabla\vec{B}$$

Hence, the drift velocity caused by the magnetic field gradient is:

$$\vec{v}_y = \frac{c}{q}\frac{\vec{F}\times\vec{B}}{B^2} = \mathrm{sign}(q)\frac{v_\perp r_L}{2}\left(\frac{\vec{B}\times\nabla\vec{B}}{B^2}\right)$$

Electrons and ions drift in opposite direction, generating electric current, which lead to particle separation and electric field in plasma. The drift velocity is independent of mass in this case.

Example: Laboratory plasma: $T = 1$ keV, $B = 10^4$ G, characteristic size $L = 1$ m; $v_d \sim 10^3$ m/s.

Curvature drift

Magnetic field lines are not necessarily straight. Particles moving along the curved magnetic lines will be affected by the centrifugal force in addition to the magnetic field gradient. We consider both of these effects in a local cylindrical coordinate system (r, θ, z).

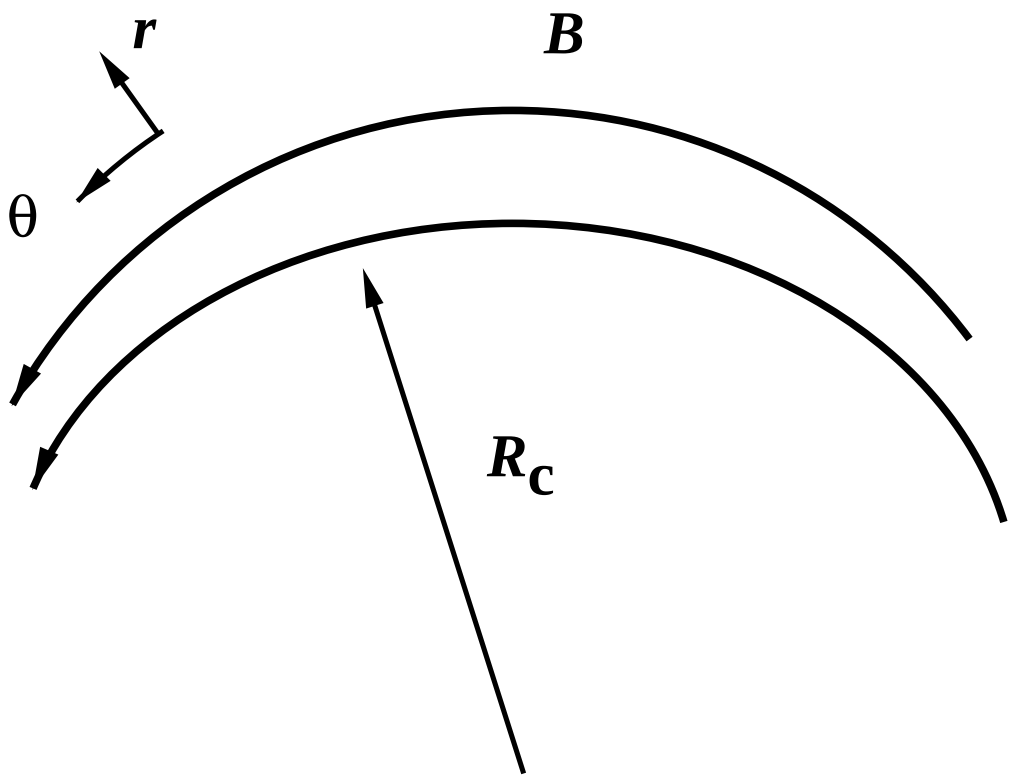

Figure 5: Non-uniform curved magnetic field.

The centrifugal force is:

$$\vec{F}_{cf} = \frac{mv_\parallel^2}{R_c}\vec{e}_r = mv_\parallel^2 \frac{\vec{R}_c}{R_c^2}$$

Hence, the curvature drift velocity is:

$$\vec{v}_{curv} = \frac{c}{q}\frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{mcv_\parallel^2}{qB^2}\frac{\vec{R}_c \times \vec{B}}{R_c^2}.$$

Let's calculate now $\nabla\vec{B}$ drift. In vacuum and low-pressure plasma $\nabla \times \vec{B} = 0$ (potential field). If B in our local coordinate system has only θ component then $\nabla\vec{B}$ has only r-component, and $\nabla \times \vec{B}$ has only z-component.

$$(\nabla \times \vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

Hence,

$$B_\theta \propto \frac{1}{r}$$

$$|\vec{B}| \propto \frac{1}{R_c}$$

$$\frac{\nabla\vec{B}}{\vec{B}} = -\frac{\vec{R}_c}{R_c^2}$$

(this can be derived more formally). The gradient drift velocity in this case is:

$$\vec{v}_{\nabla B} = -\text{sign}(q)\frac{v_\perp r_L}{2B^2}\vec{B} \times \frac{\vec{R}_c}{R_c^2}B =$$

$$= \text{sign}(q)\frac{v_\perp r_L}{2}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B} = \text{sign}(q)\frac{mcv_\perp^2}{2|q|B}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B} =$$

$$= \frac{mcv_\perp^2}{2qB}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B}$$

Then, the total drift velocity in the curved magnetic field is:

$$\vec{v}_{cf} + \vec{v}_{\nabla B} = \frac{mc}{qB}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right).$$

Since

$$\frac{|q|B}{mc} = \omega_c$$

$$\vec{v}_{cf} + \vec{v}_{\nabla B} = \text{sign}(q)\frac{\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)}{\omega_c}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B}.$$

Note that for a thermal plasma

$$v_\parallel^2 + \frac{1}{2}v_\perp^2 \propto T$$

hence

$$\vec{v}_{cf} + \vec{v}_{\nabla B} \propto \frac{T}{BR_c}.$$

This drift makes a problem for thermonuclear fusion because it increases with temperature. This can offset by increasing R_c (making bigger reactors) or B (using stronger magnets).