Collisions. Fokker-Planck Equation.

([4], pp.219-227)

We have considered the continuity equation for the distribution function, $f(\vec{r}, \vec{v}, t)$, in the 6D phase space $(\vec{r}, \vec{v})$:

$$\frac{\partial f}{\partial t} + \mathrm{div}_6(\vec{V}_6 f) = 0.$$

The distribution function represents the particle density in the 6D space. The continuity equation is obtained by considering the balance of particles in a 6D volume Ω:

$$\frac{d}{dt}\int_\Omega f d\Omega = -\oint_\Sigma f\vec{V}_6 \cdot d\Sigma$$

in the limit $\Omega \to 0$ (applying the Stokes formula to the surface integral).

However, collisions may cause significant changes in velocity. In the 6D space this looks like a particle annihilation in one place and creation in another. Therefore, in general, the effect of collisions cannot be taken into account by the divergence term. However, in the case of remote collisions velocity changes are small, and the process is diffusion like.

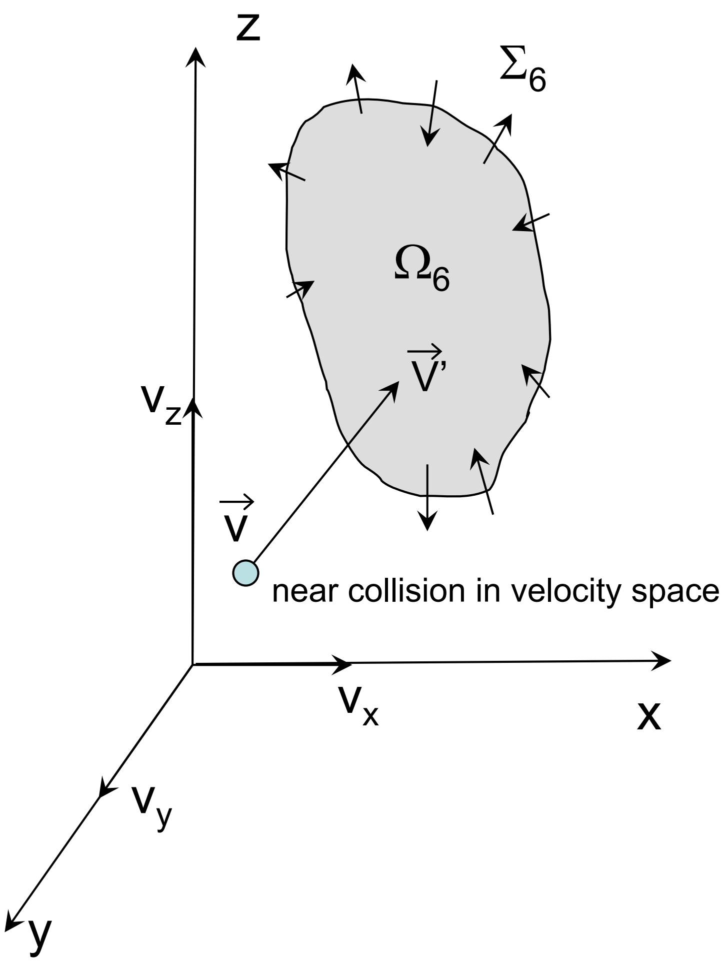

Figure 1: Collision in the 6D phase space.

Hence, by analogy with the diffusion equation:

$$\frac{\partial n}{\partial t} = \nabla(D\nabla n)$$

we can add a diffusion term in the velocity space to the kinetic equation

$$\frac{\partial f}{\partial t} + \vec{v}\nabla f + \dot{\vec{v}}\nabla_v f = \nabla_v(D\nabla_v f) \equiv \left(\frac{\partial f}{\partial t}\right)_{\rm coll}$$

This description of the collision term is called the Fokker-Planck approximation. We derive now a formula for D.

For this purpose we consider the distribution function, $f(\vec{v})$ in the velocity space ignoring the spatial variations, and define the probability $\phi(\vec{v}, \Delta\vec{v})$ for a particle of velocity $\vec{v}$ acquire an increment of velocity $\Delta\vec{v}$ in a time interval Δt.

Then, the velocity distribution at time t can be expressed through the previous velocity distribution at $t - \Delta t$:

$$f(\vec{v}, t) = \int f(\vec{v} - \Delta\vec{v}, t - \Delta t)\phi(\vec{v} - \Delta\vec{v}, \Delta\vec{v})d^3\Delta\vec{v}$$

This follows from the definition of ϕ. Also, the sum of all possible variations is:

$$\int \phi(\vec{v}, \Delta\vec{v}) d^3\vec{v} = 1.$$

In the Fokker-Planck approximation, the velocity increments are relatively small. Hence, we can consider a Taylor expansion in $\Delta\vec{v}$:

$$f(\vec{v}-\Delta\vec{v}, t-\Delta t) = f(\vec{v}, t-\Delta t) - \Delta\vec{v}\cdot\frac{\partial f(\vec{v}, t-\Delta t)}{\partial\vec{v}} +$$

$$+\frac{1}{2}\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2 f(\vec{v}, t-\Delta t)}{\partial\vec{v}\partial\vec{v}}$$

where

$$\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}} \equiv \sum_{i,j} \Delta v_i \Delta v_j \frac{\partial f}{\partial v_i \partial v_j}$$

is a diadic notation. Similarly,

$$\phi(\vec{v}-\Delta\vec{v}, \Delta\vec{v}) = \phi(\vec{v}, \Delta\vec{v}) - \Delta\vec{v}\cdot\frac{\partial\phi(\vec{v}, \Delta\vec{v})}{\partial\vec{v}} +$$

$$+\frac{1}{2}\Delta\vec{v}\Delta\vec{v} : \frac{\partial^2\phi(\vec{v}, \Delta\vec{v})}{\partial\vec{v}\partial\vec{v}}$$

Substitute these into the integral equation for

$f(\vec{v},t)$ and keep only terms up to the second order:

$$f(\vec{v},t) = f(\vec{v},t-\Delta t) - \int \Delta\vec{v} \left(\frac{\partial f}{\partial \vec{v}}\phi + \frac{\partial \phi}{\partial \vec{v}} f \right) d^3\Delta\vec{v} +$$

$$+\frac{1}{2}\int \Delta\vec{v}\Delta\vec{v} : \left(\frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}}\phi + 2\frac{\partial f}{\partial \vec{v}}\frac{\partial \phi}{\partial \vec{v}} + \frac{\partial^2 \phi}{\partial\vec{v}\partial\vec{v}} f \right) d^3\Delta\vec{v} =$$

$$= f(\vec{v},t-\Delta t) - \frac{\partial}{\partial \vec{v}} \cdot \int f\phi\Delta\vec{v} d^3\Delta\vec{v} +$$

$$+\frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \int f\phi\Delta\vec{v}\Delta\vec{v} d^3\Delta\vec{v}$$

where $f = f(\vec{v},t)$ and $\phi = \phi(\vec{v},\Delta\vec{v})$

Then, the rate of change f due to collisions:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{f(\vec{v},t) - f(\vec{v},t-\Delta t)}{\Delta t} =$$

$$= -\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{d\left\langle \Delta\vec{v} \right\rangle}{dt} f \right) + \frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \left(\frac{d\left\langle \Delta\vec{v}\Delta\vec{v} \right\rangle}{dt} f \right)$$

Here, we used that $f = f(\vec{v},t)$ is independent of $\Delta\vec{v}$, and

$$\frac{d\left\langle \Delta\vec{v} \right\rangle}{dt} = \frac{1}{\Delta t}\int \phi\Delta\vec{v} d^3\Delta\vec{v}$$

$$\frac{d\left\langle \Delta\vec{v}\Delta\vec{v} \right\rangle}{dt} = \frac{1}{\Delta t}\int \phi\Delta\vec{v}\Delta\vec{v} d^3\Delta\vec{v}.$$

The quantity $d \langle \Delta\vec{v} \rangle / dt$ is the averaged change of the mean velocity due to Coulomb collisions. Obviously, it is opposite to $\vec{v}$, and it is called "dynamic friction".

The quantities $d \langle \Delta\vec{v}\Delta\vec{v} \rangle / dt$ describe spreading particle velocities and called "velocity diffusion coefficient".

Now, we calculate tensor

$$\frac{d \langle \Delta\vec{v}\Delta\vec{v} \rangle}{dt} \equiv A_{ij}$$

for Coulomb collisions. It depends only on vector velocity $\vec{v}$, and should depend on components of this vector and . We seek this tensor in a general form:

$$A_{ij} = b \cdot \delta_{ij} + c\frac{v_i v_j}{v^2} + d \cdot \epsilon_{ijk}\frac{v_k}{v}$$

where b, c, and d are unknown coefficients, δ_{ij} and ϵ_{ijk} are the Kronecker and Levi-Civita symbols (invariant tensors). The last term is zero because the diffusion coefficient does not depend on direction of $\vec{v}$.

In order to find b and c we consider the following inner products:

$$A_{ij}\delta_{ij} = 3b + c$$

$$A_{ij}\frac{v_i v_j}{v^2} = b + c$$

because $\sum_{ij}\delta_{ij}v_i v_j/v^2 = 1$ and $\sum_{ij}\delta_{ij}\delta_{ij} = 3$.

On the other hand, for Coulomb collisions electrons with ions:

$$A_{ij}\delta_{ij} = \frac{d}{dt}\left\langle \Delta v^2 \right\rangle .$$

Since

$$(\vec{v} + \Delta\vec{v})^2 = \vec{v}^2$$

$$\Delta\vec{v}^2 = -2\vec{v}\Delta\vec{v}$$

and for Coulomb collisions:

$$\frac{d\Delta\vec{v}}{dt} = -\nu_{ei}\vec{v}$$

where ν_{ei} is the collision frequency, then

$$A_{ij}\delta_{ij} = \frac{d}{dt}\left\langle \Delta v^2 \right\rangle = 2\nu_{ei}v^2.$$

The other inner product:

$$A_{ij}\frac{v_i v_j}{v^2} = \frac{d}{dt}\frac{\left\langle v\Delta v\right\rangle^2}{v^2} \propto \Delta v^4$$

is small (we keep only up to Δv^2). Thus,

$$3b + c = 2\nu_{ei} v^2$$

$$b + c = 0$$

Hence,

$$b = \nu_{ei} v^2$$

$$c = -b$$

And, therefore,

$$A_{ij} = \nu_{ei}(v^2 \delta_{ij} - v_i v_j)$$

or,

$$\frac{d \langle \Delta\vec{v}\Delta\vec{v}\rangle}{dt} = -\nu_{ei}(\vec{I}v^2 - \vec{v}\vec{v}),$$

where I in the unit tensor. Recall that the collision frequency is

$$\nu_{ei} = n_i v \sigma_{tr} = \frac{4\pi Z^2 e^4 n_i \Lambda_c}{m^2 v^3} \equiv \frac{C}{v^3}.$$

Then, the collision term in the kinetic equation:

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} =$$

$$= C\left[\frac{\partial}{\partial \vec{v}}\left(\frac{\vec{v}f}{v^3}\right) + \frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \left(\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3} f\right)\right]$$

This equation can be simplified by using the identity:

$$\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3} \right) = -\frac{2\vec{v}}{v^3}.$$

Indeed,

$$\frac{\partial}{\partial v_j} \left(\frac{\delta_{ij} v^2 - v_i v_j}{v^3} \right) = \frac{\partial}{\partial v_j} \left(\frac{\delta_{ij}}{v} \right) - \frac{\partial}{\partial v_i} \frac{v_i v_j}{v^3} =$$

$$-\frac{1}{v^2}\frac{\partial v}{\partial v_i} - \frac{\delta_{ij} v_i v^3 + 3v^2 v v_j}{v^6} + 3\frac{v_i v_j v^2}{v^6}\frac{v_i}{v} =$$

$$= -\frac{1}{v^2}\frac{\partial v}{\partial v_i} - \frac{\delta_{ij} v_j + 3v_i}{v^3} + \frac{3v_i}{v} =$$

$$= -\frac{v_i}{v^3} - \frac{4v_i}{v^3} + \frac{3v_i}{v} = -\frac{2v_i}{v^3} = -\frac{2\vec{v}}{v^3},$$

where we used $\partial v_i/\partial v_j = \delta_{ij}$, $\partial v_j/\partial v_j = 3$, $v^2 = v_i v_i$ and $\partial v/\partial v_i = v_i/v$. Then,

$$\frac{\partial}{\partial \vec{v}} \left[\frac{\vec{v} f}{v^3} + \frac{1}{2}\frac{\partial}{\partial \vec{v}} \left(\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3} f \right) \right] =$$

$$\frac{\partial}{\partial \vec{v}} \left[\frac{\vec{v} f}{v^3} + \frac{1}{2}\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3}\frac{\partial f}{\partial \vec{v}} - \frac{\vec{v} f}{v^3} \right] =$$

$$= \frac{1}{2}\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3}\frac{\partial f}{\partial \vec{v}}.$$

Finally,

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} = \frac{C}{2}\frac{\partial}{\partial \vec{v}}\left(\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3} \cdot \frac{\partial f}{\partial \vec{v}}\right).$$

This form of the Fokker-Planck equation describes the evolution of the electron velocity distribution function $f_e(\vec{v}, t)$ due to collisions with fixed massive ions. A more general form can be derived for e-e and i-i collisions.

The plasma description in which the electrons collide only with ions and not with other electrons is called *Lorentz-gas approximation.*

Recall (Lecture 3) that the collision frequency is

$$\nu = n\sigma_{\rm tr}v$$

where $\sigma_{\rm tr}$ is the transport cross-section:

$$\sigma_{\rm tr} = \frac{4\pi Z_1^2 Z_2^2 e^4 \Lambda_C}{m^2 v^4}$$

Hence, the frequency of Coulomb e-i collisions is higher than the frequency of e-e collisions by a factor of $n_i Z^2/n_e = Z$. Thus, the Lorentz-gas approximation is reasonable for plasma with multiply charged ions.

For a Maxwellian distribution function:

$$f_e \propto \exp(-mv^2/2T)$$

the collision term vanishes. This is true for any isotropic function f, because

$$(\vec{I}v^2 - \vec{v}\vec{v}) \cdot \frac{\partial f}{\partial \vec{v}} = \underbrace{(\vec{I}v^2 - \vec{v}\vec{v})}_{0} \cdot \frac{\vec{v}}{v} \frac{\partial f}{\partial v} = 0.$$

The collision term for the Lorentz gas can be simplified:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{C}{2} \frac{\partial}{\partial \vec{v}} \left(\frac{\vec{I}v^2 - \vec{v}\vec{v}}{v^3} \cdot \frac{\partial f}{\partial \vec{v}} \right) \equiv$$

$$\equiv \frac{C}{2} \frac{\partial}{\partial \vec{v}} \left(\frac{1}{v} \vec{F} \right) = \frac{C}{2v} \frac{\partial \vec{F}}{\partial \vec{v}} - \frac{C}{2} \frac{\vec{v} \cdot \vec{F}}{v^2},$$

where

$$\vec{F} = \frac{\partial f}{\partial \vec{v}} - \frac{\vec{v}}{v^2} \left(\vec{v} \frac{\partial f}{\partial \vec{v}} \right) =$$

$$= \left(\frac{\partial f}{\partial \vec{v}}\right)_{\parallel} + \left(\frac{\partial f}{\partial \vec{v}}\right)_{\perp} - \left(\frac{\partial f}{\partial \vec{v}}\right)_{\parallel} = \left(\frac{\partial f}{\partial \vec{v}}\right)_{\perp} = \nabla_{\vec{v}_\perp} f$$

Since, $\vec{v}\cdot\vec{F}=0$,

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} = \frac{C}{2v}\frac{\partial \vec{F}}{\partial \vec{v}} = \frac{C}{2v}\nabla_{\vec{v}}\cdot\nabla_{\vec{v}_\perp} f$$

In the spherical coordinates in the velocity space:

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} = \frac{C}{2v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right].$$

(recall that $C/v^3 \equiv \nu_{ei}$.)

There is no $\partial/\partial v$ terms here because the velocity magnitude is constant in the e-i collisions. Only the direction of the velocity vector changes in these collisions.

Plasma resistivity

Suppose that the distribution is approximately Maxwellian

$$f_0 = n_e\left(\frac{m}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{mv^2}{2T_e}\right)$$

but is slight;y perturbed by a small electric field in the z direction.

The stationary Fokker-Planck equation with the first order terms is:

$$-\frac{e\vec{E}}{m}\frac{\partial f_0}{\partial \vec{v}} = \left(\frac{\partial f_1}{\partial t}\right)_{\text{coll}}$$

where f_1 is a non-Maxwellian perturbation to f_0.

The current density in the z-direction is:

$$j_z = -e\int v_z f_1 d^3v = -e\int f_1 v\cos\theta d^3v$$

(the isotropic Maxwellian distribution function gives zero contribution to the particle flux).

We find f_1 by solving the Fokker-Planck equation:

$$\frac{eEvf_0}{T}\cos\theta = \frac{C}{2v^3}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f_1}{\partial\theta}$$

where we used

$$\frac{\partial f_0}{\partial \vec{v}} = -\frac{m\vec{v}}{T}f_0.$$

We get

$$\sin\theta\frac{\partial f_1}{\partial\theta} = \frac{eEv^4 f_0}{2CT}\sin^2\theta,$$

and then

$$f_1 = -\frac{eEv^4 f_0}{CT}\cos\theta.$$

Substitute this in the equation for j_z:

$$j_z = -e \int f_1 v \cos\theta d^3 v =$$

$$= \frac{e^2 E}{CT} \int_0^\infty v^7 f_0 dv \cdot 2\pi \int_0^\pi \cos^2 \sin\theta d\theta$$

because $d^3 v = 2\pi v^2 \sin\theta d\theta dv$.

$$\int_0^\pi \cos^2\theta \sin\theta d\theta = \frac{2}{3}$$

$$\int_0^\infty v^7 f_0 dv = 48 n_e \left(\frac{m}{2\pi T}\right)^{3/2} \left(\frac{T}{m}\right)^4$$

Hence,

$$j_z = \frac{e^2 E}{CT} 2\pi \frac{32 n_e}{(2\pi)^{3/2}} \left(\frac{T}{m}\right)^{5/2} =$$

$$= \frac{16 T_e^{3/2} E}{(2\pi)^{3/2} Z e^2 m^{1/2} \Lambda_C} = \sigma E,$$

where we used

$$C = \frac{4\pi Z^2 e^4 n_i \Lambda_C}{m^2}$$

and assumed that the plasma is fully ionized:

$n_e = Zn_i$. Here

$$\sigma = \frac{16T_e^{3/2}E}{(2\pi)^{3/2}Ze^2m^{1/2}\Lambda_C}$$

is electrical conductivity. The quantity $\eta = 1/\sigma$ is plasma resistivity.

The true resistivity must include e-e collisions, and can be calculated numerically. The resulting resistivity is 1.7 times larger, it is called *Spitzer's resistivity*. The e-e collision affect the non-Maxwellian part of distribution function, reducing the number of superthermal electrons, and thus increasing the resistivity.

Runaway breakdown and electric discharges in thunderstorms

[A.V.Gurevich, K.P.Zybin, Physics - Uspekhi **44**(11), pp.1119-1140,2001]

The conventional breakdown results from the heating of electrons in an electric field. In this process, fast electrons that belong to the tail of the distribution function become able to ionize matter and, therefore, to generate new free electrons.

As soon as the electric field becomes sufficiently strong, the generation of new electrons due to ionization exceeds their disappearance due to recombination, and their number begins exponentially increasing. This phenomenon is called the electrical breakdown. The characteristic energies of the electrons responsible for ionization are 10 ± 20 eV, while recombination mostly takes place at low energies. For this reason, the mean electron energy at which breakdown occurs does not normally exceed several electron-volts. For instance, in air this energy is about 2 eV.

The runaway breakdown has an essentially different nature, which is based on the specific features of the fast particle - matter interaction. The reason is that a fast electron interacts with electrons and nuclei of neutral matter as if they were free particles, i.e., according to the Coulomb law. The Coulomb cross section is inverse proportional to the squared particle energy: $\sigma \propto 1/\epsilon^2$. That is why, the ionization braking force is $F \sim \epsilon\sigma N_m \propto 1/\epsilon$, i.e., is proportional to the molecular density N_m and inversely proportional to the electron energy, ϵ.

The decrease in the friction force is related to the possibility of the appearance of runaway electrons continuously accelerated by the field when the electric field exceeds a critical (Dreicer) field strength:

$$E > E_c.$$

The energy of the runaway electrons exceeds a critical energy ϵ_c which is determined by relativistic effects.

The decrease in the ionization braking force becomes weaker owing to relativistic effects. For $\epsilon > 1$MeV it reaches its minimum F_{min}, and then a logarithmically slow increase begins.

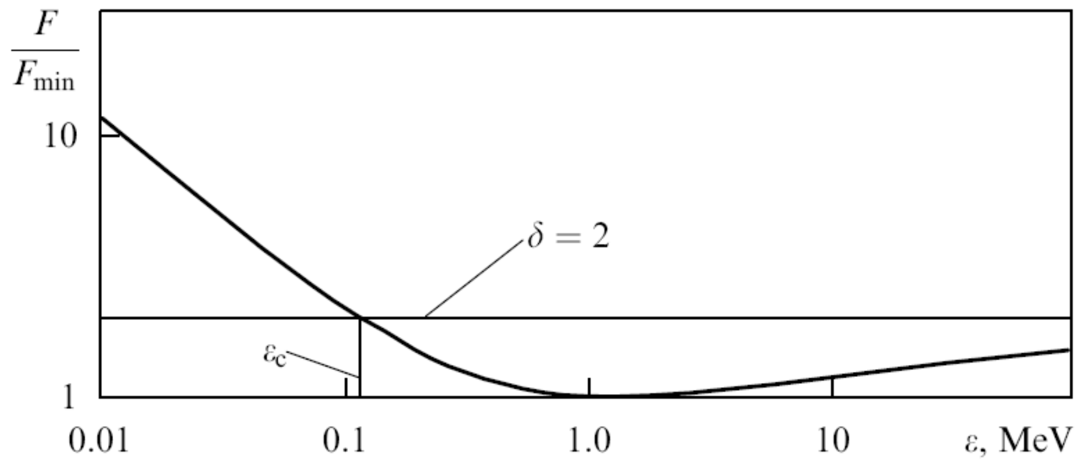

Figure 2: Dependence of the braking ionization force on the electron energy ϵ. Parameter $\delta = E/E_c$.

The runaway breakdown is related to the generation of secondary electrons due to the fast-runaway-particle ionization of neutral molecules. Although the bulk of secondary electrons have low energies, electrons with a rather high energy $\epsilon > \epsilon_c$ can also be produced. These will also become runaway electrons, i.e., they will be accelerated by the field and, in the ionization process, may in turn generate particles with $\epsilon > \epsilon_c$. As a result, an exponentially growing runaway avalanche appears.

In the non-relativistic regime:

$$F = \frac{2\pi e^4 Z N_m \Lambda_C}{\epsilon}$$

Because of the relativistic effects it reaches minimum at $\epsilon = 1.4$ MeV:

$$F_{min} = a\frac{4\pi e^4 Z N_m}{mc^2}$$

where $a \approx 11$.

In a simple elementary theory the electron motion can be described by the equation of motion:

$$m\frac{dv}{dt} = eE - F(\epsilon).$$

The more accurate theory is obtained from the kinetic Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + e\vec{E}\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{\rm coll} =$$

$$= \frac{\nu_{ei}}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\right] + \frac{1}{v^2}\frac{\partial}{\partial v}[v^2(F_D + F_B)f],$$

where the effective braking force $F = F_D + F_B$ is describes the energy loss due to ionization of molecules F_D and radiation F_B:

$$F_B = \frac{4\pi e^4 Z N_m \Lambda_C}{mv^2}$$

Numerical solution of this equation reveals exponentially growing solutions with fast particles when $\epsilon > \epsilon_c$:

$$f(v,t) = F_0(v) \exp \lambda t$$

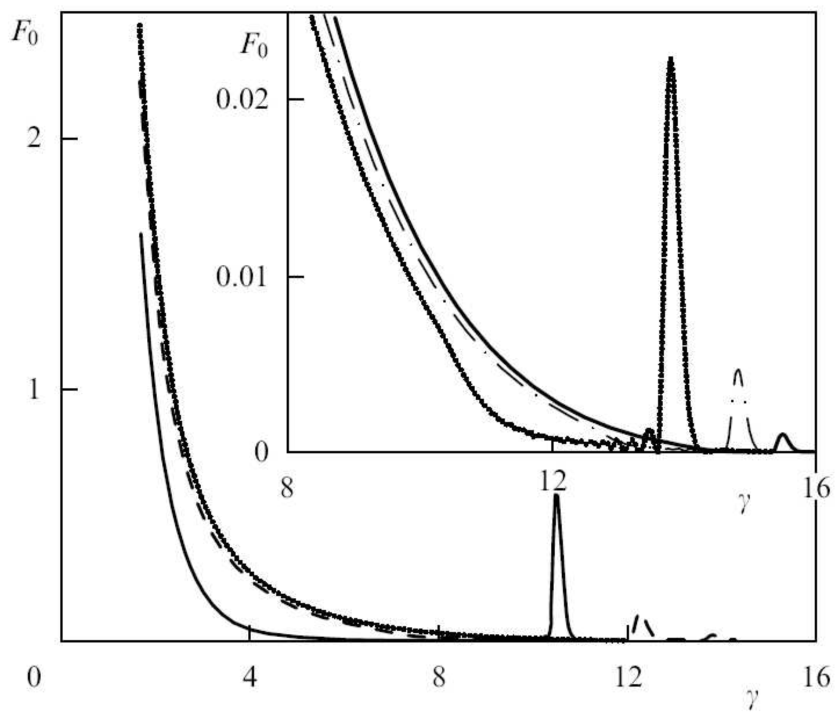

Figure 3: Function F_0 as a function of the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ for different times t: 5.6 (solid), 28 (dots), and 84 (dash-dot).

These results are applied to explain high-altitude discharges.

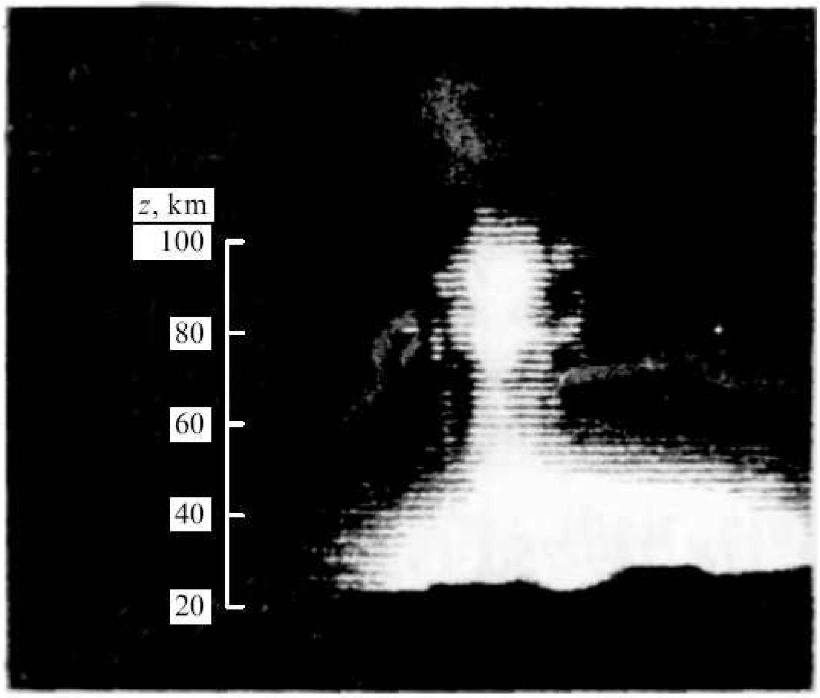

Figure 4: Example of a sprite discharge.

the discharge duration is 10 ± 200 ms, the altitudes are 25 100 km, and the horizontal extent is 10 ± 50 km. The glow-intensity peak is observed at an altitude of 50 ± 60 km. The total volume of the radiating region normally exceeds 1000 km^3.

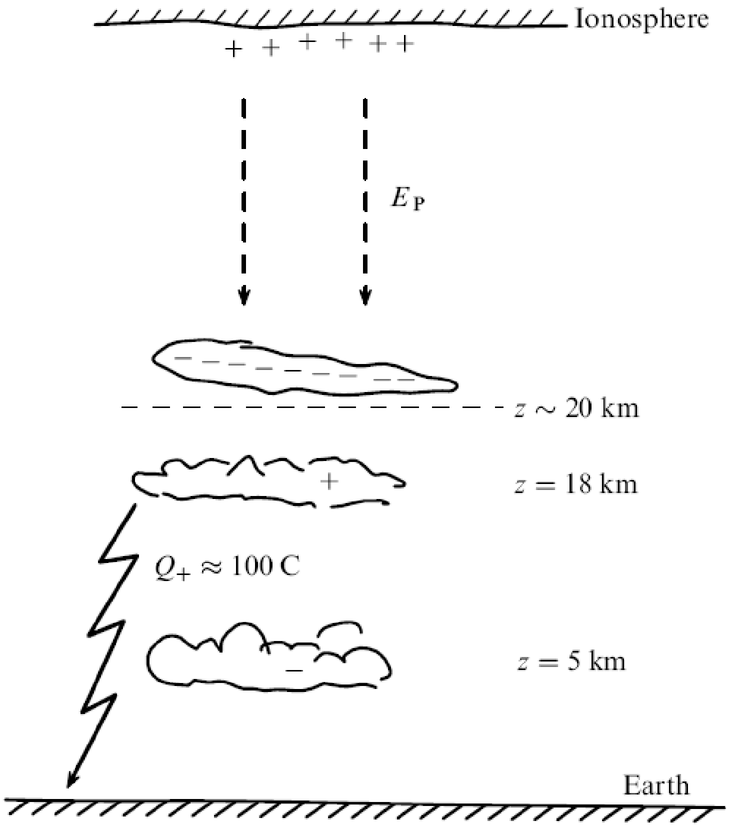

Figure 5: Model for a high-altitude discharge.