Energy and momentum transport.

Chapman-Enskog theory.

([8], p.51-75)

We derive macroscopic properties of plasma by calculating moments of the kinetic equation for particles of type a:

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial x_\beta}\left(v_\beta f_a\right) + \frac{\partial}{\partial v_\beta}\left(\frac{F_{a\beta}}{m_a} f_a\right) = St_{aa} + St_{ab}$$

where St_{aa} and St_{ab} are collision terms for $a - a$ and $a - b$ collisions.

Some properties of the collision terms result from the conservations laws:

$$\int St_{ab} d^3\vec{v} = 0$$

because there is no transformation from particles of one type into another;

$$\int m_a \vec{v} St_{aa} d^3\vec{v} = 0$$

because the momentum is conserved for collisions of particles of the same type;

$$\int \frac{m_a v^2}{2} St_{aa} d^3\vec{v} = 0$$

because of the conservation of energy for collisions of the same type particles;

$$\int m_a \vec{v} St_{ab} d^3\vec{v} + \int m_b \vec{v} St_{ba} d^3\vec{v} = 0$$

because the total momentum in $a - b$ and $b - a$ collisions is conserved;

$$\int \frac{m_a v^2}{2} St_{ab} d^3\vec{v} + \int \frac{m_b v^2}{2} St_{ba} d^3\vec{v} = 0$$

because the total energy is conserved.

We separate particle's velocity $\vec{v}$ into the mean macroscopic velocity $\vec{V}$ and fluctuation component $\vec{v}'$:

$$\vec{v} = \vec{V} + \vec{v}',$$

where

$$\vec{V} = \frac{1}{n} \int \vec{v} f d^3\vec{v} \equiv \langle \vec{v} \rangle .$$

We defined the friction force for particles a

colliding with particles b as:

$$\vec{R}_{ab} = \int m_a \vec{v}' St_{ab} d^3\vec{v}$$

Then for the conservation of the total momentum we have:

$$\vec{R}_{ab} = -\vec{R}_{ba}$$

From the conservation of the total energy we have:

$$Q_{ab} + \vec{R}_{ab}\vec{V}_a + Q_{ba} + \vec{R}_{ba}\vec{V}_b = 0$$

where

$$Q_{ab} = \int \frac{m_a v'^2}{2} St_{ab} d^3\vec{v}$$

and similarly for Q_{ba}. Then,

$$Q_{ab} + Q_{ba} = -\vec{R}_{ab}(\vec{V}_a - \vec{V}_b).$$

We consider the energy equation obtained by integrating the kinetic equation with a factor $mv^2/2$:

$$\frac{\partial}{\partial t}\left(\frac{mn}{2}\left\langle v^2\right\rangle\right) + \frac{\partial}{\partial x_\beta}\left(\frac{mn}{2}\left\langle v^2 v_\beta\right\rangle\right) - en\vec{E}\cdot\vec{V} =$$

$$= \int \frac{mv^2}{2} St_{ab} d^3\vec{v}.$$

In the local thermodynamic equilibrium, for Maxwellian distribution function:

$$\left\langle \frac{v^2}{2} \right\rangle = \frac{V^2}{2} + \left\langle \frac{v'^2}{2} \right\rangle = \frac{V^2}{2} + \frac{3T}{2m}.$$

Now, we consider the second term:

$$\left\langle \frac{v^2}{2} v_\beta \right\rangle = \sum_\alpha \left\langle \frac{(V_\alpha + v'_\alpha)^2}{2} (V_\beta + v'_\beta) \right\rangle =$$

$$= \sum_\alpha \frac{V_\alpha^2 V_\beta}{2} + V_\alpha V_\beta \underbrace{\langle v'_\alpha \rangle}_{0} + \underbrace{\frac{\left\langle {v'}_\alpha^2 \right\rangle}{2}}_{\frac{3T}{2m}} V_\beta + \frac{V^2}{2} \underbrace{\langle v'_\beta \rangle}_{0} +$$

$$+ V_\alpha \underbrace{\langle v'_\alpha v'_\beta \rangle}_{p\delta_{\alpha\beta} + \pi_{\alpha\beta}} + \frac{\left\langle {v'_\alpha}^2 v'_\beta \right\rangle}{2} =$$

$$= \left(\frac{V^2}{2} + \frac{5T}{2m} \right) V_\beta + \frac{V_\alpha \pi_{\alpha\beta}}{mn} + \frac{\left\langle v'^2 v_\beta \right\rangle}{2},$$

where we used relation $p = nT$ which follows from the definition of p.

Then, the collision term is:

$$\int \frac{m(V+v')^2}{2} St_{ab} d^3\vec{v} =$$

$$= V_\beta \int m v'_\beta St_{ab} d^3\vec{v} + \int \frac{m v'^2}{2} St_{ab} d^3\vec{v} = \vec{V}\cdot\vec{R} + Q,$$

where

$$Q = \frac{m}{2}\int v'^2 St_{ab} d^3\vec{v}.$$

Finally, the energy equation is written as

$$\frac{\partial}{\partial t}\left(\frac{nmV^2}{2} + \frac{3nT}{2}\right) +$$

$$+\frac{\partial}{\partial x_\beta}\left[\left(\frac{nmV^2}{2} + \frac{5nT}{2}\right)V_\beta + \pi_{\alpha\beta}V_\alpha + q_\beta\right] =$$

$$= en\vec{E}\cdot\vec{V} + \vec{R}\cdot\vec{V} + Q,$$

where

$$q_\beta = nm\frac{\left\langle v'^2 v'_\beta\right\rangle}{2} = \frac{m}{2}\int v'^2 v'_\beta f d^3\vec{v}.$$

Using the continuity and momentum equations we have:

$$\frac{\partial}{\partial t}\left(\frac{nmV^2}{2}\right) + \frac{\partial}{\partial x_\beta}\left(\frac{nmV^2}{2}\right) =$$

$$= mnV_\alpha \left(\frac{\partial V_\alpha}{\partial t} + V_\beta \frac{\partial V_\alpha}{\partial x_\beta} \right) =$$

$$= V_\alpha \left(-\frac{\partial p}{\partial x_\alpha} - \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} + en[\vec{E} + \frac{1}{c}\vec{V} \times \vec{B}]_\alpha + R_\alpha \right).$$

Substituting this into the energy equation we get:

$$\frac{\partial}{\partial t}\left(\frac{3nT}{2}\right) + \frac{\partial}{\partial x_\beta}\left(\frac{5nT}{2}V_\beta + \pi_{\alpha\beta}V_\alpha + q_\beta\right) -$$

$$-V_\alpha \frac{\partial p}{\partial x_\alpha} - V_\alpha \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} + enV_\alpha E_\alpha + V_\alpha R_\alpha =$$

$$= enV_\alpha E_\alpha + R_\alpha V_\alpha + Q$$

or

$$\frac{\partial}{\partial t}\left(\frac{3nT}{2}\right) + V_\beta \frac{\partial}{\partial x_\beta}\left(\frac{3nT}{2}\right) +$$

$$+\frac{5nT}{2}\frac{\partial V_\beta}{\partial x_\beta} + \pi_{\alpha\beta}\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial q}{\partial x_\beta} = Q;$$

$$\frac{3n}{2}\left(\frac{\partial T}{\partial t} + V_\beta \frac{\partial T}{\partial x_\beta}\right) + \frac{3T}{2}\underbrace{\left(\frac{\partial n}{\partial t} + \frac{\partial nV_\beta}{\partial x_\beta}\right)}_{0} +$$

$$+p\frac{\partial V_\beta}{\partial x_\beta} + \pi_{\alpha\beta}\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial q}{\partial x_\beta} = Q,$$

where we used the continuity equation and $p = nT$.

Finally, using the definition of the full derivative we get in the case of the isotropic pressure:

$$\frac{3}{2}n\frac{dT}{dt} + p\nabla \cdot \vec{V} + \nabla \cdot \vec{q} = Q,$$

where

$$\vec{q} = nm\frac{\left\langle v'^2 \vec{v}' \right\rangle}{2} = \int \frac{mv'^2}{2}\vec{v}' f d^3\vec{v}$$

is the heat flux.

So far we assumed that the local distribution function was Maxwellian. However, if the plasma is not uniform then the distribution function deviates from the Maxwellian because of mass, momentum and energy transport processes: diffusion, viscosity and heat conduction, which lead the system to the complete thermodynamic equilibrium.

The transport processes in gases were studied by Chapman and Enskog in 1916-1917, and were applied to plasma by Chapman and Cowling in 1939. The final classical transport theory was formulated by Braginskii (S.I. Braginskii, Transport processes in a plasma, in Reviews of Plasma Physics, Consultants Bureau, New York NY, 1965, Vol. 1, p.205). This theory considers

Coulomb collisions. However, particles also interact with waves and turbulence. These are considered in so-called neoclassical transport theory. Here we consider the classical theory.

The plasma fluid equations that we derived are incomplete. The viscosity tensor, $\pi_{\alpha\beta}$, heat flux, $\vec{q}$, and collision terms, $\vec{R}, Q$, can be described by equation for higher moments of the kinetic equation. However, in general this is an infinite series of moment equations because each of the moments is coupled to higher moments. Therefore, we need an additional information to close the system.

There are two basic closure schemes. In *truncation* schemes, higher order moments are assumed to vanish. For instance, one classical scheme is a "thirteen moment approximation". Another approach is to exploit a small parameter in the equations (*asymptotic* schemes). A classical example of these is the Chapman-Enskog theory, in which the small parameter is the ratio of the mean-free path between collisions to the scale length of macroscopic variations.

Consider a plasma in the local thermodynamic equilibrium described by the Maxwellian distribution function, with slowly varying macroscopic properties: $n(\vec{r},t), T(\vec{r},t), \vec{V}(\vec{r},t)$. Because of these variations the distribution function is slightly different from Maxwellian. Consider a small perturbation to the distribution function:

$$f = f_M + \delta f$$

where $\delta f \ll f_M$, and solve the Fokker-Planck equation for non-uniform plasma.

Suppose the plasma has a electron temperature gradient:

$$\nabla T_e = \vec{e}_z \frac{dT}{dz},$$

and $\vec{V} = 0$ and $n =$const, and assume that the ions are stationary because of their high mass, that there is no magnetic field, and that the plasma is isolated so that there is no current. The temperature gradient results in a pressure gradient:

$$\nabla p_e = n_e \nabla T_e$$

which is balanced by electric field created by charges on the plasma surface (this is called thermoelectric or *Seeback effect*).

The Fokker-Planck equation in this stationary case ($\partial f/\partial t$ has the form:

$$v_z \frac{\partial f}{\partial z} - \frac{e}{m} E_z \frac{\partial f}{\partial v_z} = \frac{C}{2v^3} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial f}{\partial\theta},$$

where

$$C = \frac{4\pi Z^2 e^4 n_i \Lambda_C}{m^2}$$

(recall that $C/v^3 = \nu_{ei}$ is the collision frequency.

We substitute $f = f_M + \delta f$ into this equation. In the right-hand side, the term with f_M vanishes because when $\vec{V} = 0$,

$$f_M = n_e \left(\frac{m}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{mv^2}{2T_e}\right)$$

does not depend on angle θ in the velocity space. In the left-hand side we can neglect δf compared to f_M.

Thus, we get the following equation for δf:

$$v_z \frac{\partial f_M}{\partial z} - \frac{e}{m} E \frac{\partial f_M}{\partial v_z} = \frac{C}{2v^3} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial \delta f}{\partial\theta}$$

Calculating the derivatives:

$$\frac{\partial f_M}{\partial v_z} = -\frac{mv_z}{T_e} f_M = -\frac{mv\cos\theta}{T_e} f_M$$

$$\frac{\partial f_M}{\partial z} = -\frac{3}{2T_e}\frac{\partial T_e}{\partial z} f_M + \frac{mv^2}{2T_e^2}\frac{\partial T_e}{\partial z} f_M =$$

$$= \frac{1}{2T_e}\frac{\partial T_e}{\partial z}\left(\frac{mv^2}{T_e} - 3\right) f_M,$$

we get

$$\frac{v\cos\theta}{2T_e}\frac{\partial T_e}{\partial z}\left(\frac{mv^2}{T_e} - 3\right) f_M + \frac{eEv\cos\theta}{T_e} f_M =$$

$$= \frac{C}{2v^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\delta f}{\partial\theta}.$$

We seek the solution of this equation in the form:

$$\delta f = \Phi(v)\cos\theta$$

and get:

$$\Phi(v) = -\frac{v^3}{C}\left[\frac{v}{2T_e}\frac{\partial T_e}{\partial z}\left(\frac{mv^2}{T_e} - 3\right) + \frac{eEv}{T_e}\right] f_M.$$

The distribution function must satisfy the stationary condition (mean velocity equals zero):

$$\int \delta f \vec{v} d^3\vec{v} = 0$$

that is

$$\int \Phi(v) v \cos^2\theta d^3\vec{v} = 0,$$

or

$$\int_0^\infty \Phi(v) v d^3 v = 0,$$

where $d^3v = 4\pi v^2 dv$. This gives

$$\frac{1}{2}\frac{\partial T_e}{\partial z}\left(\frac{m}{T_e}\left\langle v^7\right\rangle - 3\left\langle v^5\right\rangle\right) - eE\left\langle v^5\right\rangle = 0,$$

where

$$\left\langle v^5\right\rangle = \int f_M v^5 d^3 v =$$

$$= 4\pi n_e \left(\frac{m}{2\pi T_e}\right)^{3/2} \int_0^\infty \exp\left(-\frac{mv^2}{2T_e}\right) v^7 dv =$$

$$= 4\pi n_e \left(\frac{m}{2\pi T_e}\right)^{3/2} \left(\frac{2T_e}{m}\right)^4 \int_0^\infty e^{-x^2} x^7 dx =$$

$$= 48\sqrt{\frac{2}{\pi}} n_e \left(\frac{T_e}{m}\right)^{5/2}.$$

(we used $\int_0^\infty \exp(-x^2) x^{2k+1} dx = k!/2$)

Similarly,

$$\left\langle v^7\right\rangle = \int f_M v^7 d^3 v = 384\sqrt{\frac{2}{\pi}} n_e \left(\frac{T_e}{m}\right)^{7/2}.$$

Then, we have

$$\frac{1}{2}\frac{\partial T_e}{\partial z}(384 - 3 \cdot 48) = 48eE.$$

This gives a relation for the electric field strength which provides the plasma equilibrium

$$eE = \frac{5}{2}\frac{\partial T_e}{\partial z}.$$

Now, we can consider the momentum equation for electrons and find the force $\vec{R}_{ei}$ that acts on electron from ions:

$$-\nabla p_e - n_e eE + R_{ei} = 0$$

Since

$$\nabla p_e = n_e \frac{\partial T_e}{\partial z}$$

we get

$$R_{ei} = -\frac{3}{2} n_e \frac{\partial T_e}{\partial z}.$$

This is called *thermoforce.* It appears because the collision frequency depends on velocity. Electrons which come from higher temperature regions experience less friction than electrons which come from lower temperature regions. The net result is a macroscopic friction force directed against the temperature gradient.

Using the solution for δf we can calculate the heat flux $\vec{q}$:

$$\vec{q} = \int \frac{mv^2}{2} \vec{v} f d^3\vec{v}$$

$$q_z = 2\pi \int \frac{mv^3}{2} \Phi(v) \cos^2\theta \sin\theta d\theta v^2 dv$$

Using the expression for $\Phi(v)$ with already determined E,

$$\Phi(v) = -\frac{v^4}{2CT_e}\left[\frac{mv^2}{T_e} + 8\right]\frac{\partial T_e}{\partial z} f_M,$$

we get

$$q_z = -\frac{m}{4CT_e}\frac{\partial T_e}{\partial z}\frac{1}{3}\left(\frac{m}{T_e}\left\langle v^9\right\rangle - 8\left\langle v^7\right\rangle\right)$$

where

$$\left\langle v^9\right\rangle = \int f_M v^9 d^3v = 3840\sqrt{\frac{2}{\pi}}\left(\frac{T_e}{m}\right)^{9/2} n_e.$$

Thus,

$$q_z = -\frac{m}{12CT_e}\frac{\partial T_e}{\partial z}(3840 - 8\cdot 384)\sqrt{\frac{2}{\pi}}\left(\frac{T_e}{m}\right)^{7/2} n_e =$$

$$= -64\sqrt{\frac{2}{\pi}}\frac{n_e}{C}\left(\frac{T_e}{m}\right)^{5/2}\frac{\partial T_e}{\partial z} =$$

$$= -8\sqrt{\frac{2}{\pi}}\frac{T_e^{5/2}}{m^{1/2}e^4 Z\Lambda_C}\frac{\partial T_e}{\partial z},$$

where we used the expression for the collision parameter C and $n_e = Zn_i$.

Generally, the heat flux equation is

$$\vec{q} = -\kappa_e \nabla T_e,$$

where

$$\kappa_e = 8\sqrt{\frac{2}{\pi}}\frac{T_e^{5/2}}{m^{1/2}e^4 Z\Lambda_C}$$

is the thermal conductivity for electrons. It depends only on temperature.

Consider an order-of-magnitude estimate:

$$\kappa \sim \frac{n_e}{\nu_{ei} v^3}\left(\frac{T_e}{m}\right)^{5/2} \sim n_e \tau_{ei}\frac{T_e}{m},$$

where $\tau_{ei} = 1/\nu_{ei}$ is the characteristic collision time.

In the case of the stationary plasma we have the heat conduction equation:

$$\frac{3}{2}n_e\frac{\partial T_e}{\partial t} = -\nabla \cdot \vec{q} = \nabla(\kappa \nabla T_e)$$

In 1D case,

$$\frac{\partial T_e}{\partial t} = \chi\frac{\partial^2 T_e}{\partial z^2},$$

where

$$\chi = \frac{\kappa}{3/2n_e} \sim \frac{T_e}{m}\tau_{ei} \sim v_T^2\tau_{ei} \sim \lambda_{ei}v_T,$$

is the temperature diffusion coefficient, λ_{ei} is the mean-free path, v_T is the electron thermal velocity.

Order-of-magnitude estimate for the deviation from the Maxwellian distribution:

$$\delta f/f_M \sim \Phi(v)/f_M \sim \frac{v_T}{\nu_{ei}}\frac{\partial \ln T_e}{\partial z} \sim \lambda_{ei}/L,$$

where $L = (\partial \ln T_e/\partial z)^{-1}$ is the characteristic length scale of plasma temperature variations. Therefore, the Chapman-Enskog theory is valid when $\lambda \ll L$.

Consider a simple solution to the heat conduction equation for an instant point release of energy Q at

$t = 0$ and $x = 0$:

$$\frac{3}{2}n\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial z^2} + \epsilon\delta(z)\delta(t)$$

where $\epsilon = 3/2nQ$ and

$$Q = \int_{-\infty}^{\infty} T dz.$$

The solution is well-known:

$$T = \frac{Q}{(4\pi\chi t)^{1/2}} \exp\left(-\frac{z^2}{4\chi t}\right)$$

The heat expands as $l \sim \sqrt{\chi t}$ with infinite speed. This is obviously unphysical.

Another difficulty is that when the temperature gradient is very high the heat flux calculated as $\vec{q} = -\kappa\nabla T$ may exceed the maximum possible heat flux that can be transported by all electrons moving with thermal velocity. This phenomenon is called heat flux saturation. The saturation heat flux is usually estimated as

$$q_{\rm sat} \sim \alpha \cdot nT \cdot v_T$$

where coefficient $\alpha \sim 1/6$. Thus, one practical recipe is

$$q = max\left(q_{\rm sat}, \left|\kappa \frac{dT}{dz}\right|\right).$$

The reason for this difficulty is that the stationary condition of energy transport is not satisfied for strong heat fluxes, and thus one cannot neglect the time derivative in the kinetic equation:

$$\frac{\partial \delta f}{\partial t} + v_z \frac{\partial f_M}{\partial z} - \frac{e}{m} E \frac{\partial f_M}{\partial v_z} = St[\delta f]$$

Applying the same theory we now get the heat-flux equation in the form:

$$\tau \frac{\partial q}{\partial t} - \kappa \frac{\partial T}{\partial z} = q$$

Combining this with the energy equation:

$$\frac{3}{2} n \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial z}$$

we get a new equation for temperature:

$$\frac{3}{2} n \tau \frac{\partial^2 T}{\partial t^2} + \frac{3}{2} n \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2},$$

or

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial z^2}.$$

This equation provides a finite speed of heat conduction $\sim \sqrt{\chi/\tau} \sim \alpha v_T$, where $\alpha < 1$ is a coefficient (sometimes taken as 1/6).

It is remarkable that the heat flux equations with the time derivative was first derived by Maxwell in 1867.

and neglecting terms of the forms $\xi\eta$ and ξ^3 and $\xi\eta^2$ when not multiplied by the large coefficient k_1, we find by equations (75), (77), and (54),

$$\left.\begin{array}{l}\rho\frac{\partial}{\partial t}\beta(\xi^3+\xi\eta^2+\xi\zeta^2)+\beta\frac{d}{dx}\cdot\rho(\xi^4+\xi^2\eta^2+\xi^2\zeta^2)-\beta(\xi^2+\eta^2+\zeta^2)\frac{dp}{dx}-2\beta\xi^2\frac{dp}{dx}\\ \quad =-3k_1\rho^2A_2\beta\{\xi^3+\xi\eta^2+\xi\zeta^2\}.\end{array}\right\} \qquad (143)$$

The first term of this equation may be neglected, as the rate of conduction will rapidly establish itself. The second term contains quantities of four dimensions in ξ, η, ζ, whose values will depend on the distribution of velocity among the molecules. If the distribution of velocity is that which we have proved to exist when the system has no external force acting on it and has arrived at its final state, we shall have by equations (29), (31), (32),

$$\overline{\xi^4}=3\overline{\xi^2}.\overline{\xi^2}=3\frac{p^2}{\rho^2}, \qquad (144)$$

$$\overline{\xi^2\eta^2}=\overline{\xi^2}.\overline{\eta^2}=\frac{p^2}{\rho^2}, \qquad (145)$$

$$\overline{\xi^2\zeta^2}=\overline{\xi^2}.\overline{\zeta^2}=\frac{p^2}{\rho^2}; \qquad (146)$$

and the equation of conduction may be written

$$5\beta\frac{p^2}{\rho\theta}\frac{d\theta}{dx}=-3k_1\rho^2A_2\beta\{\xi^3+\xi\eta^2+\xi\zeta^2\}. \qquad (147)$$

[Addition made December 17, 1866.]

Figure 1: Maxwell, J.C., On the dynamical theory of gases, Phil.Trans. Roy. Soc.Lon., 1867, v.157, 49-88. The first term of Eq.143 corresponds to $\tau\partial q/\partial t$, ξ, η, ζ are the three components of particle velocity, $(\xi^3+\xi\eta^2+\xi\zeta^2)\propto q$.

Finally, we consider the two-fluid and single-fluid energy equations:

$$\frac{3}{2}n_e\frac{dT_e}{dt} + p_e\nabla\vec{V}_e + \nabla\vec{q}_e = Q_{ei}$$

$$\frac{3}{2}n_i\frac{dT_i}{dt} + p_i\nabla\vec{V}_i + \nabla\vec{q}_i = Q_{ie}$$

The single-fluid approximation is obtained by taking the sum of these equations:

$$\sum_{ei}\frac{3}{2}n_a\frac{dT_a}{dt} + p_a\nabla\vec{V}_a + \nabla\vec{q}_a = Q_{ei} + Q_{ie}.$$

This is reduced to

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla\left(\frac{3}{2}p\vec{V}\right) + p\nabla\vec{V} = -\nabla\vec{q} + \vec{u}\nabla p_e + \sum Q,$$

where $p = p_e + p_i$, $\vec{V} = \vec{V}_i$ (we neglect electron inertia in the single-fluid approximation),
$\vec{u} = \vec{V}_e - \vec{V}_i$, $\vec{q} = \vec{q}_e + \vec{q}_i + \frac{5}{2}p_e\vec{u}$,

$$\sum Q = Q_{ei} + Q_{ie} = -\vec{R}_{ei}(\vec{V}_e - \vec{V}_i).$$

We recall that the friction force can be expressed in terms of the collision frequency:

$$\vec{R}_{ei} = m_e n_e(\vec{V}_i - \vec{V}_e)\overline{\nu}_{ei}$$

and that the electric current density is:

$$\vec{j} = en_e(\vec{V}_i - \vec{V}_e),$$

and find:

$$\sum Q = \frac{m_e \overline{\nu}_{ei}}{e^2 n_e} \vec{j}^2 = \frac{\vec{j}^2}{\sigma},$$

where

$$\sigma = \frac{e^2 n_e}{m_e \overline{\nu}_{ei}}$$

is the electrical conductivity. Thus, $\sum Q$ gives the Joule heating. This term depends on the collision frequency and does not change for plasma in magnetic field.

The terms in Q_{ei} and Q_{ie} that are canceled in the sum describe temperature relaxation:

$$Q_{ie} = \frac{3m_e}{m_i} \frac{n_e(T_e - T_i)}{\tau_{ei}}$$

$$Q_{ei} = -\frac{3m_e}{m_i} \frac{n_e(T_e - T_i)}{\tau_{ei}} + \vec{j}^2/\sigma,$$

where τ_{ei} is the electron collision time.