Plasma transport in magnetic field.

Ambipolar diffusion.

([4], p.185-217; Chen, p.68-70, 190-195)

We have considered the process of energy transport in plasma, and found that it is caused by fast electrons. The heat flux for relatively small temperature gradients is given by

$$\vec{q} = -8\left(\frac{2}{\pi}\right)^{3/2}\frac{T_e^{5/2}}{m^{1/2}e^4 Z\Lambda_C}\nabla T_e.$$

We have discussed two basic ways to describe transport processes in plasma:

1. the Chapman-Enskog asymptotic method of solving the kinetic equation for small deviations to the Maxwellian distribution function;
2. using higher moments of the kinetic equations.

However, these methods have some algebraic complexity. Now we consider effects of magnetic field on the transport processes using a qualitative approach.

Consider first 1D particle diffusion for a density profile:

$$n(x) = n(x_0) + \left.\frac{\partial n}{\partial x}\right|_{x=x_0} \cdot (x - x_0)$$

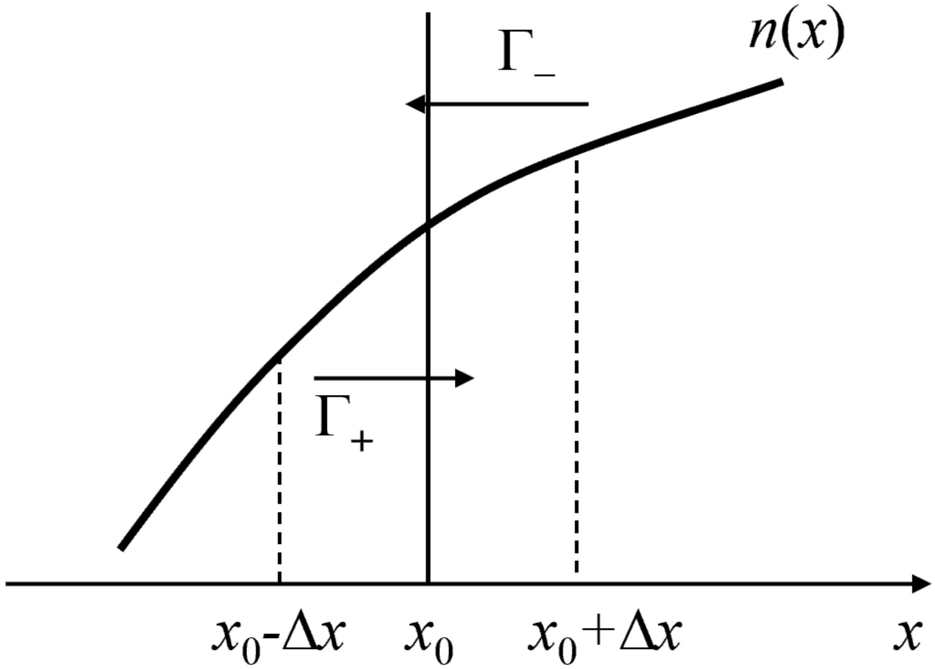

Figure 1:

Assuming that between collisions each particle can move to distance Δx with equal probability in both directions along the x-axis, we estimate the particle flux at point $x = x_0$. The number of particles making the rightward step from $x_0 - \Delta x$ to $x = x_0$

in unit time (positive flux) is

$$\Gamma_+ = \underbrace{\frac{1}{2}}_{probability} \cdot \frac{1}{\Delta t} \cdot \underbrace{\int_{x_0-\Delta x}^{x_0} n(x)dx}_{number\ of\ particles\ to\ the\ left}$$

Similarly, the negative flux is:

$$\Gamma_- = -\frac{1}{2}\frac{1}{\Delta t}\int_{x_0}^{x_0+\Delta x} n(x)dx.$$

Substituting the equation for $n(x)$ we get:

$$\Gamma_+ = \frac{1}{2\Delta t}\int_{x_0-\Delta x}^{x_0}\left(n(x_0) + (x - x_0)\left.\frac{dn}{dx}\right|_{x_0}\right) dx =$$

$$= \frac{1}{2\Delta t}\left(n\Delta x - \frac{(\Delta x)^2}{2}\frac{dn}{dx}\right);$$

$$\Gamma_- = -\frac{1}{2\Delta t}\left(n\Delta x + \frac{(\Delta x)^2}{2}\frac{dn}{dx}\right).$$

The net flux is:

$$\Gamma = \Gamma_+ + \Gamma_- = -\frac{(\Delta x)^2}{2\Delta t}\frac{dn}{dx} = -D\frac{dn}{dx},$$

where

$$D = \frac{(\Delta x)^2}{2\Delta t}.$$

Consider the number of particle change in $[x_0 - \Delta x, x_0 + \Delta x]$:

$$\int_{x_0-\Delta x}^{x_0+\Delta x} \Delta n dx = [\Gamma(x_0 - \Delta x) - \Gamma(x_0 + \Delta x)]\Delta t$$

or for small Δx:

$$\Delta n \cdot (2\Delta x) = -\frac{\partial \Gamma}{\partial x}(2\Delta x)\Delta t$$

and finally:

$$\frac{\partial n}{\partial t} = -\frac{\partial \Gamma}{\partial x}.$$

Combining this with

$$\Gamma = -D\frac{\partial n}{\partial x}$$

we get the diffusion equation:

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial n}{\partial x}\right).$$

The diffusion coefficient can be estimated as:

$$D \approx \frac{\lambda^2}{\tau} = \nu\lambda^2 \sim v_T\tau,$$

where λ is the free mean path, τ is the collision time, $\nu = 1/\tau$ is the collision frequency, and v_T is the thermal velocity.

However, displacement between collisions is not necessarily equal to the mean-free path,

$$\Delta x \neq \lambda.$$

This is the case for magnetized plasma.

Consider a plasma in a strong magnetic field. Particle motions along the magnetic field lines are not affected by magnetic field. Therefore, the diffusion coefficient along the field lines is the same as in the case without magnetic field, that is

$$D_{\|} \sim \nu \lambda^2 \sim \frac{v_T^2}{\nu_{ei}} \sim \frac{T}{m\nu_{ei}}.$$

The diffusion coefficient is inverse proportional to the collision frequency. Thus, collisions reduce transport. However, for diffusion perpendicular to the magnetic lines collisions enhance transport.

Without collisions there is no transport across the field lines. Of course, we know that there are drift motions across the field lines. However, drifts are often arranged in the form of closed orbits within a bounded plasma, and don't carry particles outside the plasma.

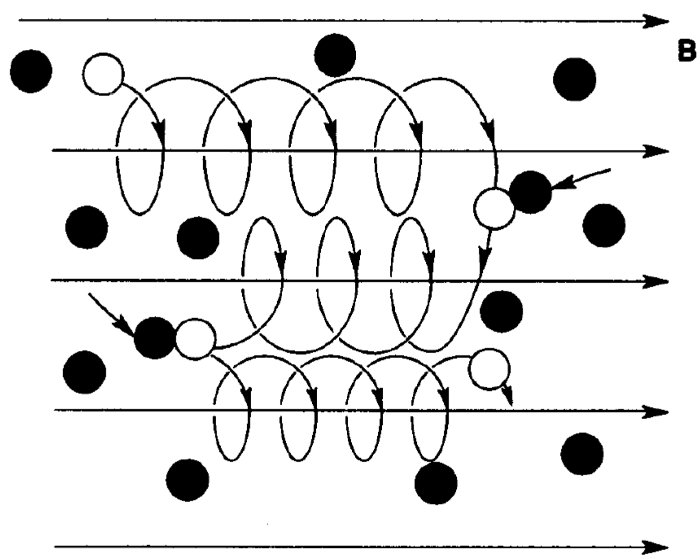

Figure 2: Diffusion of a charged particle (open circle) in magnetic field due to collisions with other particles (full circles).

However, when there are collisions particles can migrate randomly across the field lines. Collisions change locations of gyration ("guiding") centers. If the collision frequency is much smaller than the gyration frequency:

$$\nu_{ei} \ll \nu_c \equiv \frac{\omega_c}{2\pi}$$

then particles will do full Larmor orbits between collisions then the step size in the random walk is equal to the Larmor radius,

$$\Delta x \sim r_L.$$

Thus, the diffusion coefficients perpendicular to magnetic field is:

$$D_\perp \sim \nu_{ei} r_L^2,$$

where

$$r_L = v_T/\omega_c = \frac{v_T mc}{eB}.$$

Diffusion in fully ionized plasmas

In a fully ionized plasma, diffusion perpendicular to the magnetic field lines can be considered in the single-fluid approximation:

$$\frac{\partial \rho}{\partial t} + \nabla(\vec{v}\rho) = 0,$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{\vec{j} \times \vec{B}}{c},$$

$$\eta \vec{j} = \vec{E} + \frac{\vec{v} \times \vec{B}}{c} + \frac{\nabla p_e}{ne} - \frac{\vec{j} \times \vec{B}}{nec},$$

where $p = p_e + p_i$.

The last equation is the generalized Ohm's law, and collisions between electrons and ions appear as the resistivity term.

We consider slow processes and ignore inertia $\rho \frac{d\vec{v}}{dt}$ in the momentum equation.

Thus, we have the force balance equation:

$$\frac{\vec{j} \times \vec{B}}{c} = \nabla p.$$

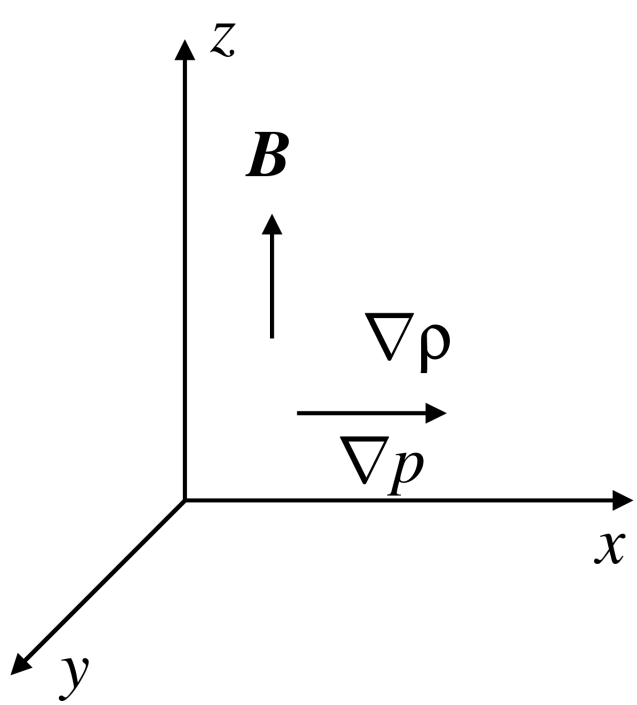

Figure 3:

Suppose that $\vec{B} = (0, 0, B)$, $\vec{j} = (j_x, j_y, 0)$ and $\rho = \rho(x)$. Then, from the force balance equation we get

$$j_x = 0$$

$$j_y = \frac{c}{B}\frac{dp}{dx}$$

Since there is no variation in the y direction, the electric currents automatically satisfy the quasi-neutrality condition:

$$\nabla \cdot \vec{j} = 0.$$

The electric field is only in the x direction because the plasma is uniform in other directions.

From the Ohm's law we get:

$$E_x + \frac{v_y B}{c} + \frac{1}{ne}\frac{dp_e}{dx} - \frac{j_y B}{nec} = 0$$

$$\eta j_y = -\frac{v_x B}{c}$$

From these equations we find the perpendicular components of velocity:

$$v_x = -\frac{\eta c}{B} j_y = -\frac{\eta c^2}{B^2}\frac{dp}{dx}$$

$$v_y = -\frac{E_x c}{B} - \frac{c}{neB}\frac{dp_e}{dx} + \frac{c}{neB}\frac{dp}{dx}$$

or

$$v_y = \underbrace{-\frac{E_x c}{B}}_{E\ drift} + \underbrace{\frac{c}{neB}\frac{dp_i}{dx}}_{diamagnetic\ drift}$$

This equation describes the electric and diamagnetic drifts is y direction. There is no diffusion in this direction because the plasma is uniform along the y axis.

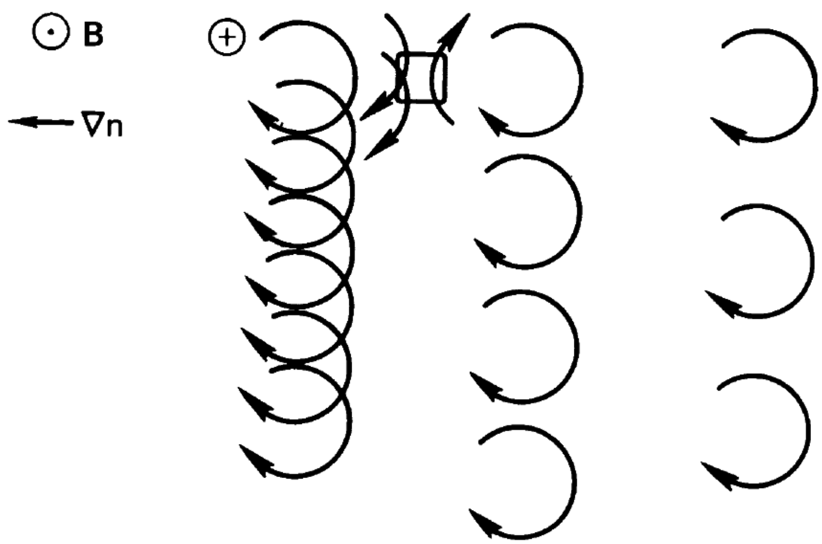

Figure 4: Origin of diamagnetic drift: because of the density gradient more ions moving downward than upward, since the downward moving ions come from a region of higher density.

We note that the electric drift is the same as for individual particles (Lecture 4). However, so-called diamagnetic drift exists only in plasma fluids. It is caused by the density gradient. The drifts do not cause diffusion.

The second equation for v_x describes the diffusion:

$$v_x = -\frac{\eta c^2}{B^2}\frac{dp}{dx}.$$

This is a resistivity driven flow antiparallel to the pressure gradient. In a vector form

$$\vec{v}_\perp = -\frac{\eta c^2}{B^2}\nabla_\perp p$$

Substituting this into the continuity equation we get

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\rho\eta c^2}{B^2}\frac{\partial p}{\partial x}\right)$$

For isothermal plasma $p = \rho T/M$:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{p\eta c^2}{B^2}\frac{\partial\rho}{\partial x}\right)$$

Hence, the perpendicular diffusion coefficient is:

$$D_\perp = \frac{p\eta c^2}{B^2}$$

It is called *classical diffusion coefficient*. It is the same for electrons and ions. This is called *ambipolar diffusion.*

Since the electrical resistivity is related to the collision frequency as

$$\eta = \frac{m\nu_{ei}}{ne^2}$$

and $p = n(T_e + T_i)$, we obtain the estimate:

$$D_\perp \sim \frac{\nu_{ei} c^2 m (T_e + T_i)}{e^2 B^2} \sim \nu_{ei} r_{Le}^2 \left(1 + \frac{T_i}{T_e}\right)$$

where

$$r_{Le} = \frac{m v_T c}{eB} = \frac{\sqrt{m T_e} c}{eB}$$

is the Larmor radius for electrons.

Thus the cross-field diffusion is a random walk of electrons with step r_{Le} and frequency of steps ν_{ei}.

Fully-ionized plasma is intrinsically ambipolar. Electrons and ions diffuse at the same rate because of quasineutrality. The intrinsic ambipolarity is the consequence of the conservation of momentum in ion-electron collisions.

To understand more deeply the diffusion results we consider collisions between particles of the same and different types.

For particles of the same type there is no diffusion because their guiding centers after a collision move in the opposite direction across the field lines. Only collisions of unlike particles, electrons and ions, give rise to diffusion.

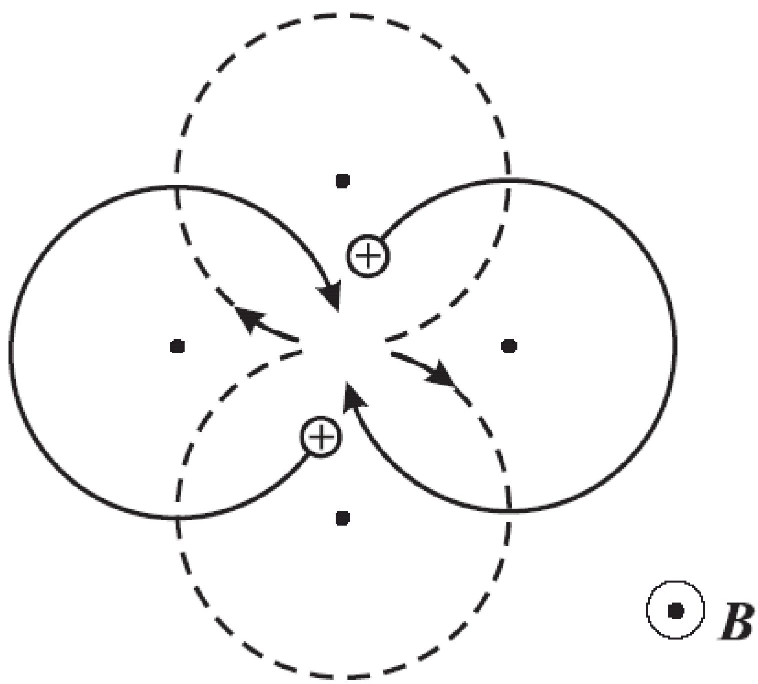

Figure 5: Displacement of the guiding centers in a 90-degree collision of two ions.

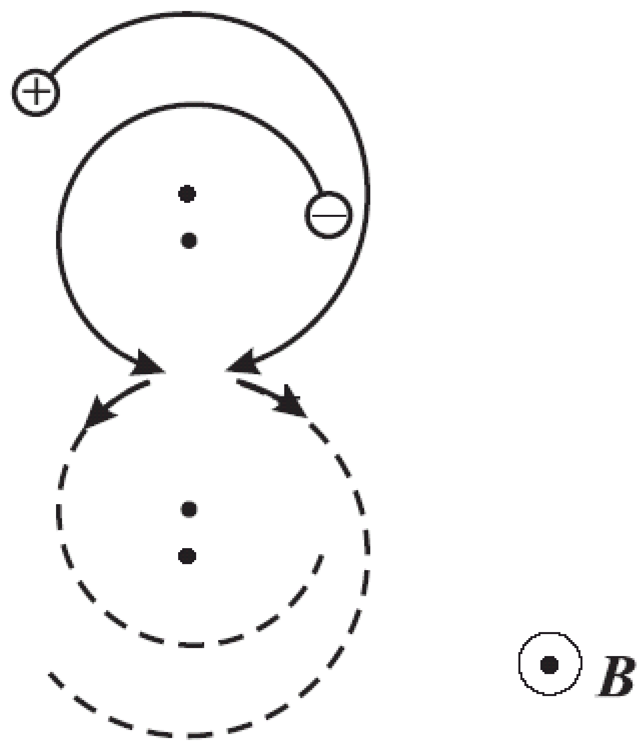

Figure 6: Displacement of the guiding centers in a head-on collision of electrons and ions.

Heat transport in magnetic field

Without magnetic field the heat transported mostly by electrons:

$$\vec{q}_e = -\kappa \nabla T_e$$

where

$$\kappa \sim \frac{n v_{Te}^2}{\nu_{ei}} \sim n\lambda v_{Te}$$

where the mean free path λ is approximately the same for electrons and ions. However, the electron thermal velocity is approximately $\sqrt{M/m}$ times higher than the ion velocity. Hence, the ion heat heat conduction coefficient is $\sqrt{M/m}$ times smaller the electron one. Since the transport along the magnetic field lines is the same as without magnetic field, the heat transport along the field lines is mostly carried by electrons.

Consider the temperature diffusion coefficients for electrons and ions across the field lines:

$$\chi_e \sim r_{Le}^2 \nu_e$$

$$\chi_i \sim r_{Li}^2 \nu_{ii}$$

Since $r_L = mvc/eB$ and $\nu = nv\sigma_{\rm tr} \propto 1/(m^2 v^3)$:

$$\frac{\chi_e}{\chi_i} = \frac{v_{Ti}}{v_{Te}} = \sqrt{\frac{m}{M}}$$

that is across the magnetic field the heat conduction coefficient for ions is higher than for electrons.

Neo-classical transport

We have found that the transport coefficient across the magnetic field lines are inverse proportional to B^2:

$$D_\perp = \frac{\eta c^2 p}{B^2} \sim r_{Le}^2 \nu_{ei}$$

However, experiments showed that

$$D_\perp \propto \frac{1}{B}.$$

Bohm found a semiempirical formula that fits the experiments:

$$D_\perp = \frac{1}{16}\frac{cT_e}{eB}$$

so-called Bohm diffusion. It represents the maximum diffusion rate in plasma perpendicular to the magnetic field lines.

This can be explained by plasma instabilities. Plasma waves or inhomogeneities give rise to a collisionless random walk process, in which the particle flux is determined by random fluctuating electric fields, ($\vec{E} \times \vec{B}$ drifts):

$$\Gamma_\perp \sim n v_\perp \sim n \frac{E}{B},$$

where E can estimated from energy equipartition:

$$eE \cdot L \sim T_e$$

where L is a characteristic scale.

If $n/L \sim \gamma \nabla n$, then

$$\Gamma_\perp \sim \frac{T_e}{eB} \frac{n}{L} \sim \gamma \frac{T_e}{eB} \nabla n$$

For $\gamma = 1/16$ we obtain the Bohm formula.

Alternative explanation.

Plasma waves increase the collision frequency. Since

$$D_\perp \sim r_{Le}^2 \nu$$

the coefficient $D_\perp$ increases. When ν is higher than the cyclotron frequency $\omega_c/2\pi$ then the plasma will behave as without magnetic field, that is

$$D \sim \frac{v_T^2}{\nu}.$$

This means that the diffusion coefficient will decrease with further growth of ν. Hence, the

maximum diffusion rate is when $nu \sim \omega_c/2\pi$.
Thus,

$$D_\perp \sim r_{Le}^2 \omega_c/2\pi \sim \frac{m^2 v_{Te}^2 c^2}{e^2 B^2} \frac{eB}{mc} \sim \frac{1}{2\pi} \frac{cT_e}{eB}.$$