Electromagnetic waves in plasma.

(Chen, F.F., Introduction to Plasma Physics and Controlled Fusion, 1984, Ch.4.12-4.17, p.114-135)

Electromagnetic waves in plasma without magnetic field

Propagation of electromagnetic waves in plasma is described by Maxwell equations:

$$\nabla \times \vec{B}_1 = \frac{1}{c}\frac{\partial \vec{E}_1}{\partial t} + \frac{4\pi}{c}\vec{j}_1$$

$$\nabla \times \vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{B}_1}{\partial t}$$

$$\nabla \cdot \vec{E}_1 = 4\pi\rho$$

$$\nabla \cdot \vec{B}_1 = 0$$

Here we use index 1 for waves and 0 for background fields. We consider first the case of zero magnetic field $\vec{B}_0 = 0$.

Consider propagation along z axis:

$$\vec{B}_1, \vec{E}_1 \propto \exp\left(i\vec{k}\vec{r} - i\omega t\right)$$

We consider plane waves.

Then, we obtain the following system of linear equations:

$$i\vec{k} \times \vec{B}_1 = -\frac{i\omega}{c}\vec{E}_1 + \frac{4\pi}{c}\vec{j}_1$$

$$i\vec{k} \times \vec{E}_1 = \frac{i\omega}{c}\vec{B}_1$$

If we consider high-frequency waves, light or microwaves, than the ions can be considered as fixed. Then, the current comes only from electron motion:

$$\vec{j}_1 = -en_0\vec{v}_1$$

From the equation of motion

$$m\frac{d\vec{v}_1}{\partial t} = -e\vec{E}_1$$

we have

$$\vec{v}_1 = \frac{e\vec{E}_1}{im\omega}$$

Then we obtain equation for $\vec{E}_1$:

$$\vec{B}_1 = \frac{c}{\omega}\vec{k} \times \vec{E}_1$$

$$\vec{k} \times (\vec{k} \times \vec{E}_1)\frac{ic}{\omega} = -\frac{i\omega}{c}\vec{E}_1 - \frac{4\pi}{c}\frac{n_0 e^2}{im\omega}\vec{E}_1$$

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = -\frac{\omega^2}{c^2}\vec{E}_1 + \frac{4\pi n_0 e^2}{c^2 m}\vec{E}_1$$

or

$$\vec{k} \cdot (\vec{k} \cdot \vec{E}_1) - k^2\vec{E}_1 = -\frac{\omega^2}{c^2}\vec{E}_1 + \frac{\omega_p^2}{c^2}\vec{E}_1$$

where

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m}$$

is the plasma frequency. If we take the inner product of $\vec{k}$ and the first equation

$$\vec{k} \cdot \left(\vec{k} \times \vec{B}_1 = -\frac{i\omega}{c}\vec{E}_1 - \frac{4\pi}{c}\frac{n_0 e^2}{im\omega}\vec{E}_1\right)$$

in the left-hand side we obtain

$$\vec{k} \cdot (\vec{k} \times \vec{B}_1) = 0$$

because

$$\nabla\vec{B}_1 = i\vec{k} \cdot \vec{B}_1 = 0$$

Then, from the right-hand side we obtain:

$$\vec{k} \cdot \vec{E}_1 = 0$$

Hence, the electromagnetic waves are transverse ($\vec{k}$ and $\vec{E}_1$ are perpendicular).

Using this, we finally obtain the wave dispersion relation

$$-k^2 \vec{E}_1 = -\frac{\omega^2}{c^2} \vec{E}_1 + \frac{\omega_p^2}{c^2} \vec{E}_1$$

$$\frac{\omega^2}{c^2} = k^2 + \frac{\omega_p^2}{c^2}$$

or

$$\omega^2 = k^2 c^2 + \omega_p^2$$

This is the dispersion relation for electromagnetic waves in plasma without magnetic field. The plasma frequency plays the role of cutoff frequency. Waves with frequencies lower than ω_p cannot propagate in plasma. This is used for plasma diagnostics to measure plasma density. Also, this phenomenon is important for propagation of radio waves in the ionosphere, shortwave radio communications.

Phase speed $v_\phi \equiv \omega/k$ is higher than the speed of light. The refraction index: $n = c/v_\phi = ck/\omega < 1$. Thus a concave plasma lens is convergent. This is used for plasma heating.

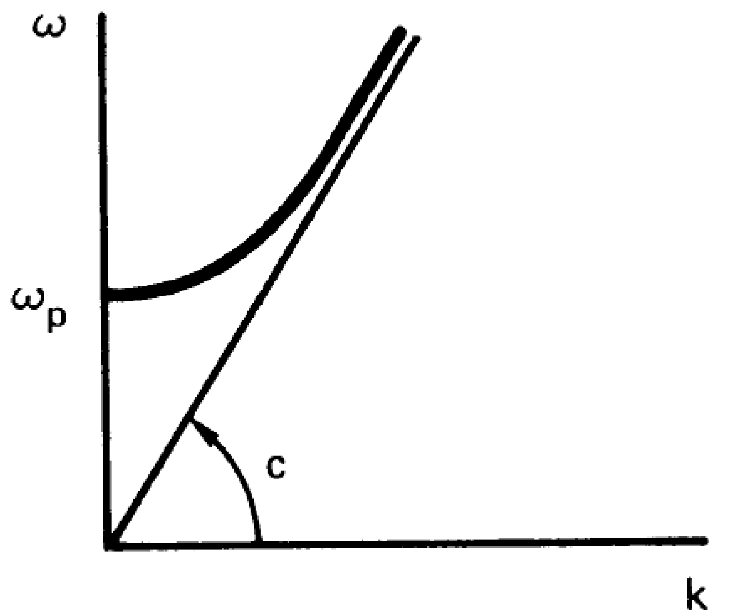


Figure 1: Dispersion relation for electromagnetic waves in plasma without magnetic field.

Below the cutoff frequency the signal decays:

$$kc = \sqrt{\omega^2 - \omega_p^2} = i\sqrt{\omega_p^2 - \omega^2}$$

$$E_1 \propto \exp(ikx) \sim \exp(-x/\delta)$$

where $\delta = |k|^{-1} = c/\sqrt{\omega_p^2 - \omega^2}$ is skin depth.

Electromagnetic waves in plasma with magnetic field

Consider now electromagnetic waves in plasma with magnetic field. There two general types of electromagnetic waves: ordinary waves in which electrons move along the magnetic field lines, and thus are not affected by the magnetic field, and extraordinary waves that are affected by magnetic field.

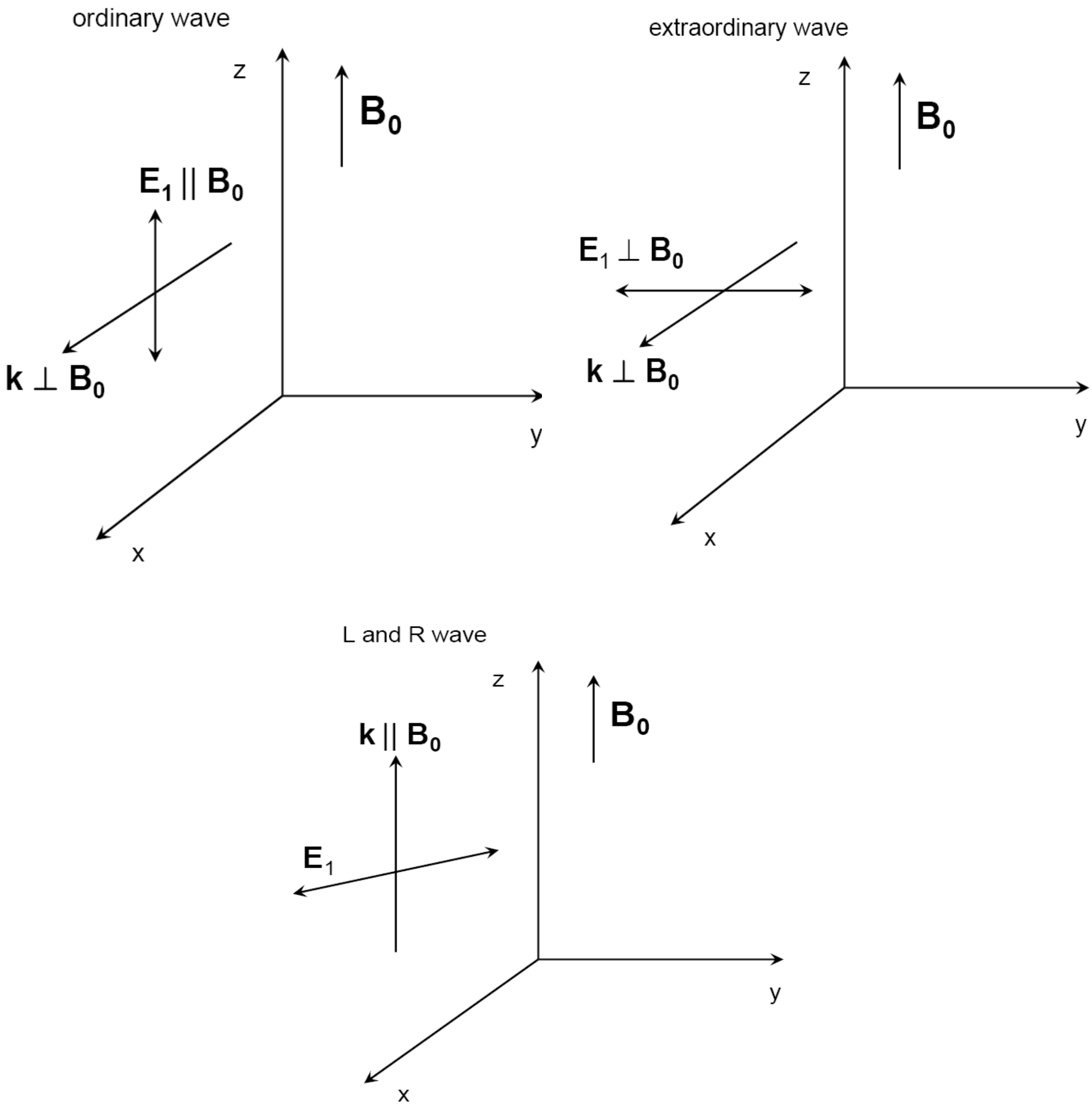


Figure 2: Classification of electromagnetic waves in plasma with magnetic field.

In the first case of ordinary waves: $\vec{E}_1 \parallel \vec{B}_0$ and $\vec{k} \perp \vec{B}_0$.

Consider, for instance, the case when

$$\vec{B}_0 = (0, 0, B)$$

$$\vec{E}_1 = (0, 0, E_1)$$

$$\vec{k} = (k, 0, 0)$$

Then

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \vec{E}_1 = \frac{4\pi i}{\omega} \vec{j}_1$$

Since $\vec{E}_1 = E\vec{e}_z$ we have to consider only z component of the equation of electron motion:

$$m\frac{dv_z}{dt} = -eE_1$$

Then we obtain exactly the same relation as without the magnetic field:

$$\omega^2 = \omega_p^2 + k^2c^2$$

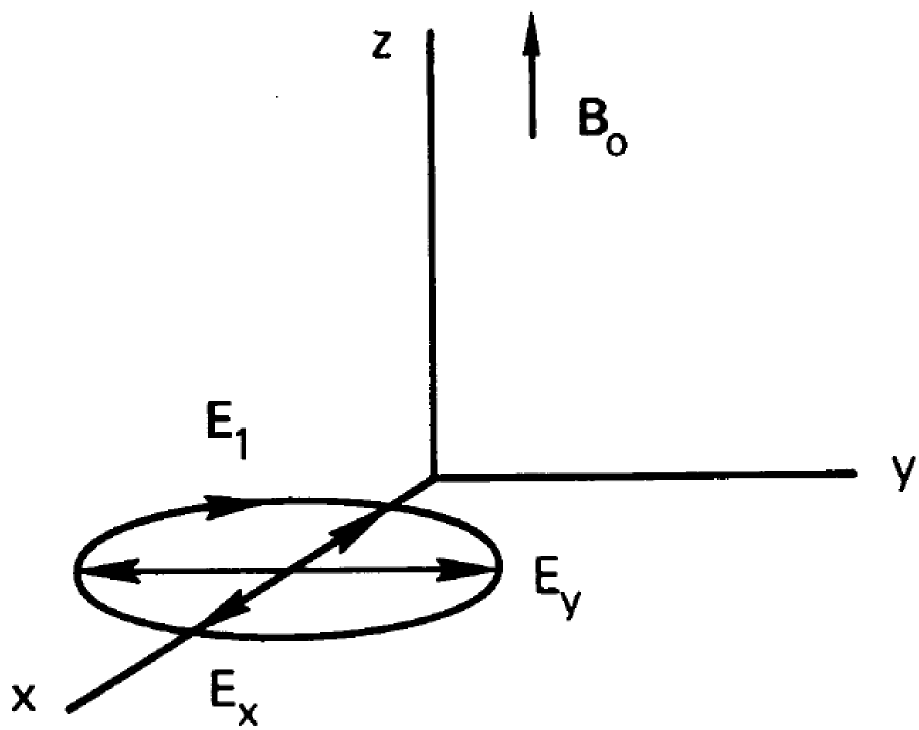


Figure 3: The $\vec{E}$-vector of an extraordinary wave is elliptically polarized. The components E_x and E_y oscillate 90° out of phase, so that the total vector rotates along the ellipse.

Consider now a case of extraordinary (L and R) waves: $\vec{E}_1 \perp \vec{B}_0$ and $\vec{k} \parallel \vec{B}_0$:

$$\vec{B}_0 = (0, 0, B)$$

$$\vec{E}_1 = (E_x, E_y, 0)$$

$$\vec{k} = (0, 0, k)$$

Then the equation of motion is

$$m\frac{d\vec{v}}{dt} = -e\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}_0\right]$$

$$-im\omega\vec{v} = -e\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}_0\right]$$

For simplicity we dropped index "1"

Since

$$\vec{v} \times \vec{B}_0 = \vec{e}_x v_y B_0 - \vec{e}_y v_x B_0$$

we obtain

$$v_x = -\frac{ie}{m\omega}\left(E_x + \frac{v_y}{c}B_0\right)$$

$$v_y = -\frac{ie}{m\omega}\left(E_y - \frac{v_x}{c}B_0\right)$$

We find v_x and v_y from this linear system:

$$v_x + \frac{ieB_0}{m\omega c} v_y = -\frac{ie}{m\omega} E_x$$

$$-\frac{ieB_0}{m\omega c} v_x + v_y = -\frac{ie}{m\omega} E_y$$

$$v_x \left(1 - \frac{\omega_c^2}{\omega^2}\right) = -\frac{ei}{m\omega} \left(E_x - \frac{ieB_0}{m\omega c} E_y\right)$$

$$v_x = -\frac{ie}{m\omega} \frac{E_x - (\omega_c/\omega) E_y i}{1 - \omega_c^2/\omega^2}$$

$$v_y = -\frac{ie}{m\omega} \frac{E_y + (\omega_c/\omega) E_x i}{1 - \omega_c^2/\omega^2}$$

where

$$\omega_c = \frac{eB_0}{mc}$$

is the electron cyclotron frequency.

Then, we substitute these into the Maxwell equations:

$$\vec{k} \times (\vec{k} \times \vec{E}) = -\frac{\omega^2}{c^2} \vec{E} + \frac{4\pi\omega}{ic^2} \vec{j}$$

$$\vec{k}(\vec{k}\vec{E}) - k^2 \vec{E} = -\frac{\omega^2}{c^2} \vec{E} + \frac{4\pi\omega}{ic^2} \vec{j}$$

since we have $\vec{k}\vec{E} = 0$ hence

$$\left(\frac{\omega^2}{c^2} - k^2\right)\vec{E} = \frac{4\pi\omega}{ic^2}\vec{j}$$

or

$$(\omega^2 - c^2k^2)\vec{E} = i4\pi n_0 e\omega\vec{v}$$

Then, using the equations for v_x and v_y:

$$(\omega^2 - c^2k^2)E_x = \frac{4\pi n_0 e^2}{m}\frac{1}{1 - \frac{\omega_c^2}{\omega^2}}\left(E_x - i\frac{\omega_c}{\omega}E_y\right)$$

or

$$(\omega^2 - c^2k^2)E_x = \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}}\left(E_x - i\frac{\omega_c}{\omega}E_y\right)$$

Introducing parameter

$$\alpha = \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}}$$

we obtain equations for E_x and E_y and the dispersion relation

$$(\omega^2 - c^2k^2 - \alpha)E_x + i\alpha\frac{\omega_c}{\omega}E_y = 0$$

$$(\omega^2 - c^2k^2 - \alpha)E_y - i\alpha\frac{\omega_c}{\omega}E_x$$

$$(\omega^2 - c^2k^2 - \alpha)^2 - \alpha^2 \frac{\omega_c^2}{\omega^2} = 0$$

Solving this equation we obtain:

$$\omega^2 - c^2k^2 - \alpha = \pm\alpha \frac{\omega_c}{\omega}$$

or

$$\omega^2 - c^2k^2 = \alpha \left(1 \pm \frac{\omega_c}{\omega}\right) =$$

$$= \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} \left(1 \pm \frac{\omega_c}{\omega}\right) = \frac{\omega_p^2}{1 \mp \frac{\omega_c}{\omega}}$$

Thus, we obtained two solutions:

1. R-wave
$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega}$$
2. L-wave
$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 + \omega_c/\omega}$$

To understand the physical sense of these solutions we have to look at the solution for E_x and E_y:

for the R-wave we obtain:

$$\alpha \frac{\omega_c^2}{\omega^2}(E_x + iE_y) = 0$$

or

$$E_x + iE_y = 0$$

and for the L-wave we have:

$$E_x - iE_y = 0$$

The R and L waves are circularly polarized.

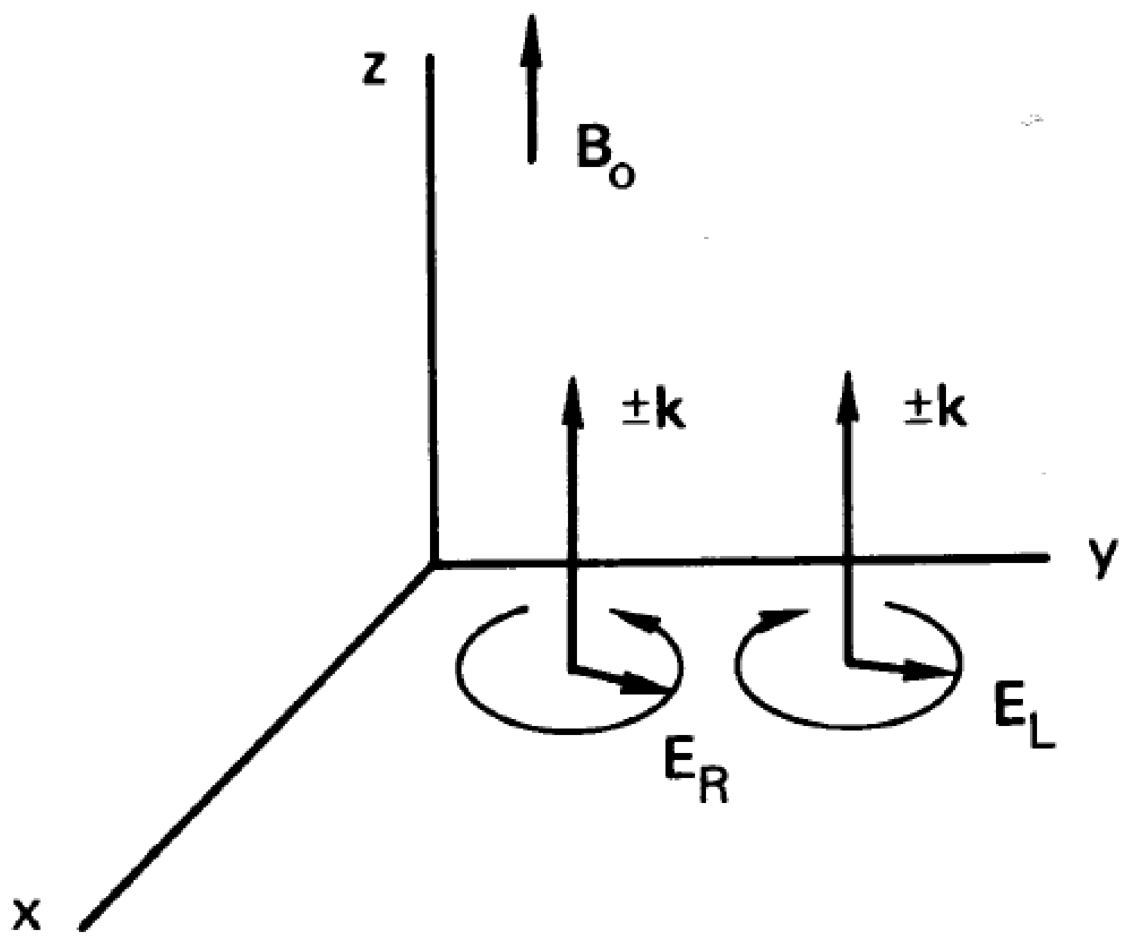


Figure 4: Geometry of circularly polarized L- and R-waves propagating along B_0.

For instance, for the R-wave:

$$E_y = -iE_x$$

We define

$$E_x = E_0 \Re[\exp(-i\omega t + ikz)] = E_0 \cos(-\omega t + kz)$$

then

$$E_y = E_0 \Re[i \exp(-i\omega t + ikz)] = -E_0 \sin(-\omega t + kz)$$

For $z = 0$ the electric field oscillates as:

$$E_x = E_0 \cos(\omega t)$$

$$E_y = E_0 \sin(\omega t)$$

Similarly, for the L-wave, but in the opposite direction. For R wave vector $\vec{E}$ rotates in the same direction as electron gyration. Thus, there is resonance at $\omega = \omega_c$. For L waves the directions are opposite - no resonance.

Consider the phase-speed $v_\phi \equiv \omega/k)$ diagram for R and L waves.

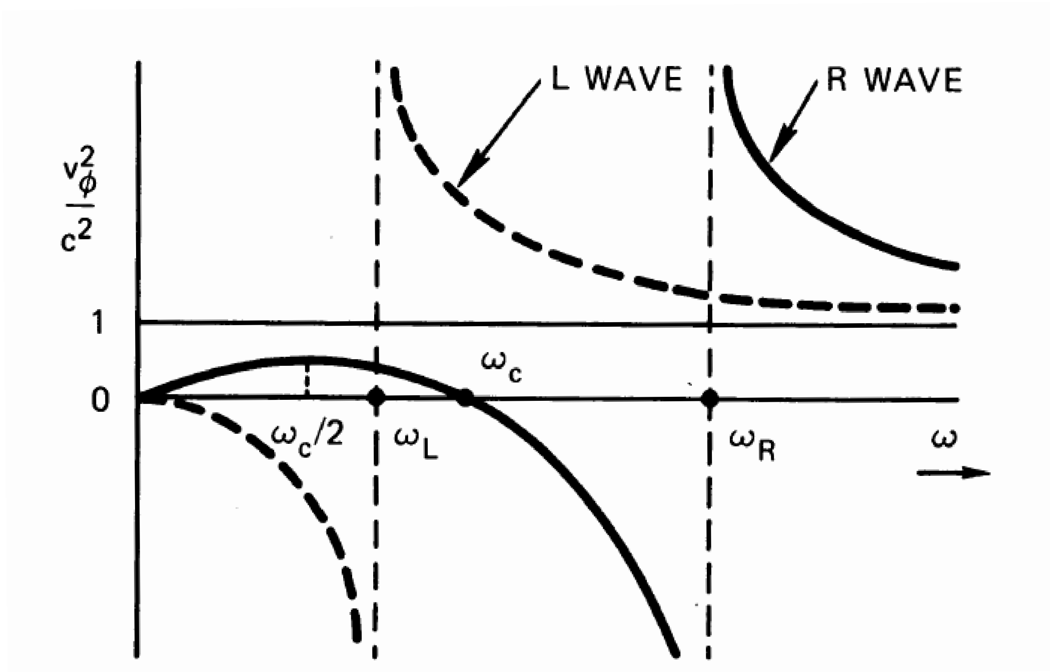


Figure 5: The phase speed ($v_\phi \equiv \omega/k$) diagram for L and R waves. The regions of $v_\phi^2 < 0$ are the regions of nonpropagation.

R wave:

$$\frac{\omega^2}{c^2k^2} = \frac{1}{1 - \frac{\omega_p^2}{\omega(\omega-\omega_c)}} = \frac{\omega(\omega - \omega_c)}{\omega^2 - \omega\omega_c - \omega_p^2}$$

L wave:

$$\frac{\omega^2}{c^2k^2} = \frac{1}{1 - \frac{\omega_p^2}{\omega(\omega+\omega_c)}} = \frac{\omega(\omega + \omega_c)}{\omega^2 + \omega\omega_c - \omega_p^2}$$

Waves propagate when $k^2 > 0$, and don't propagate if $k^2 < 0$. Thus, the cutoff frequencies are determined at $k = 0$:

- R waves:

$$1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)} = 0$$

$$\omega^2 - \omega\omega_c - \omega_p^2 = 0$$

  we find the cutoff frequency ω_R from this quadratic equation:

$$\omega_R = \frac{\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}}{2}$$

- L waves:

$$1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)} = 0$$

$$\omega^2 + \omega\omega_c - \omega_p^2 = 0$$

$$\omega_L = \frac{-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}}{2}$$

Consider the dispersion relation L and R waves.

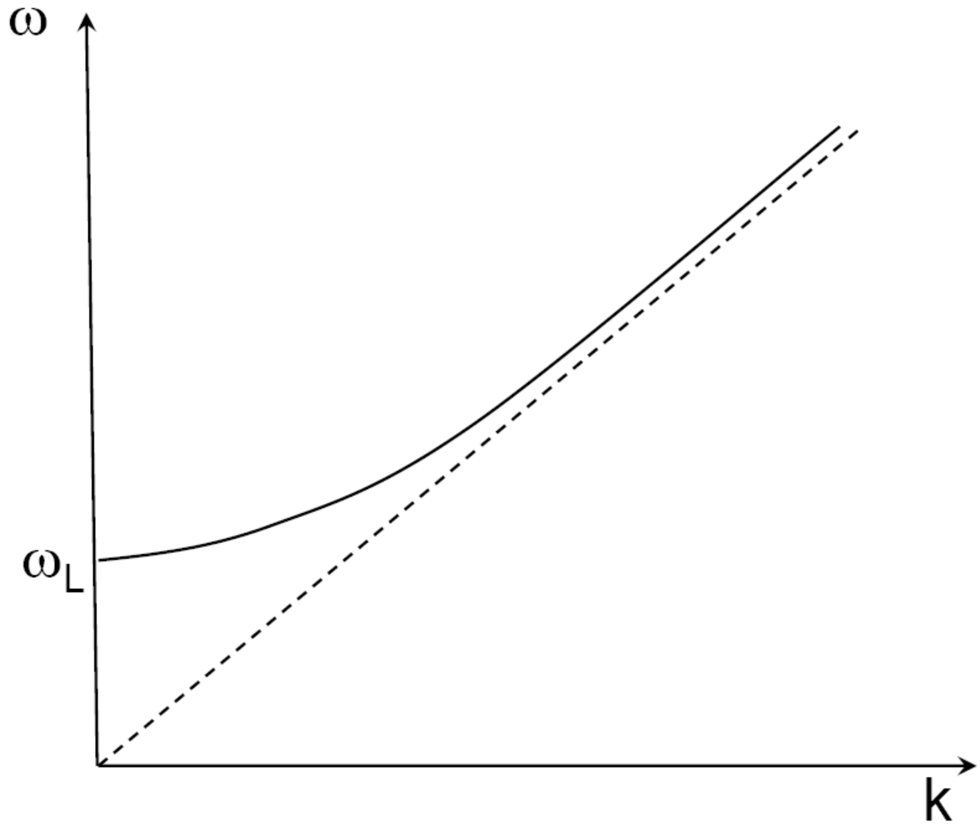


Figure 6: The dispersion relation for L waves. ω_L is the cutoff frequency.

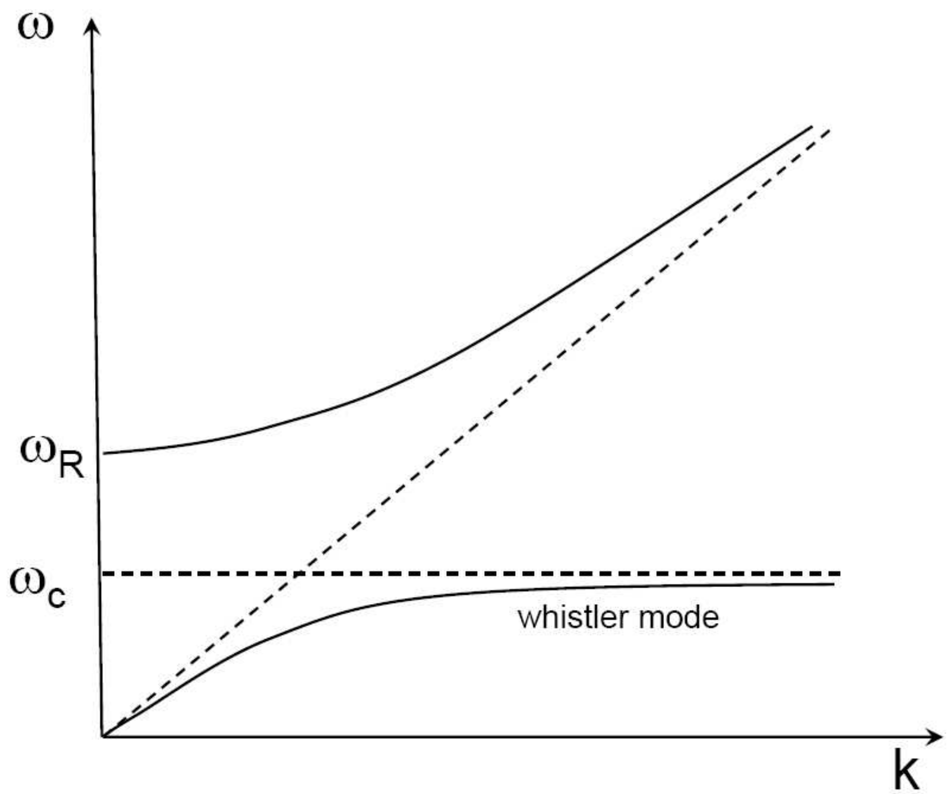


Figure 7: The dispersion relation for R waves. ω_R is the cutoff frequency. ω_c is the cyclotron frequency.

For L waves the dispersion relation is similar to the dispersion relation without magnetic field.

Whistler mode

For R waves there is additional low-frequency mode at $\omega < \omega_c$: *whistler mode* (electron cyclotron wave) which is very important ionospheric phenomenon.

In whistler mode the direction of rotation of the polarization vector is the same as the direction of gyration of electrons. Hence, the wave accelerate electrons and cannot propagate at $\omega = \omega_c$.

At small k the dispersion relation is:

$$\frac{c^2k^2}{\omega^2} \sim \frac{\omega_p^2}{\omega\omega_c}$$

or

$$\omega \sim \frac{c^2\omega_c}{\omega_p^2}k^2$$

or group velocity

$$\frac{\partial\omega}{\partial k} \propto k \propto \sqrt{\omega}$$

Thus the low frequencies arrive later, giving rise to the descending tone.

Several whistlers can be produced by a single lightning flash because of propagation along different magnetic flux tubes.

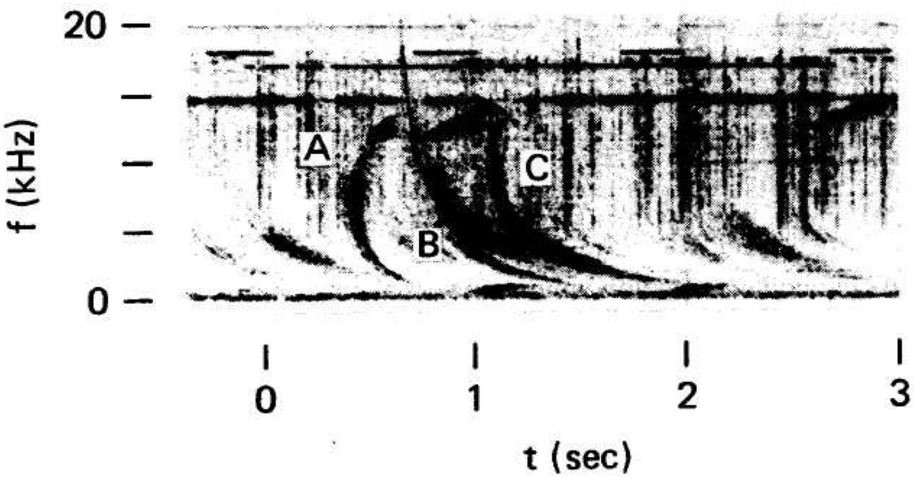


Figure 8: Actual spectrogram of whistler signal. The downward motion of the dark curves indicates a descending glide tone.

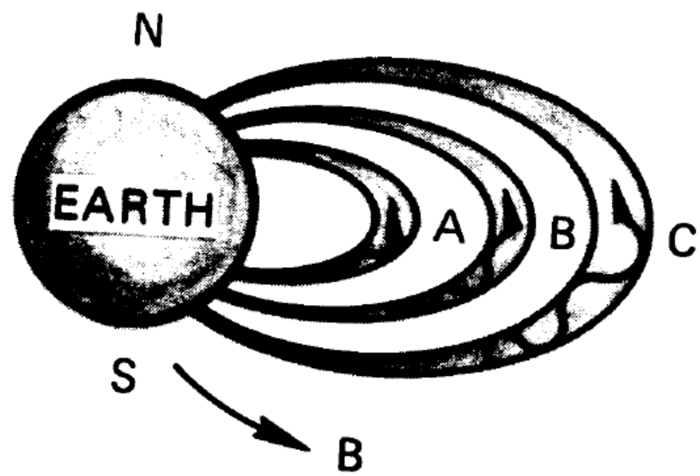
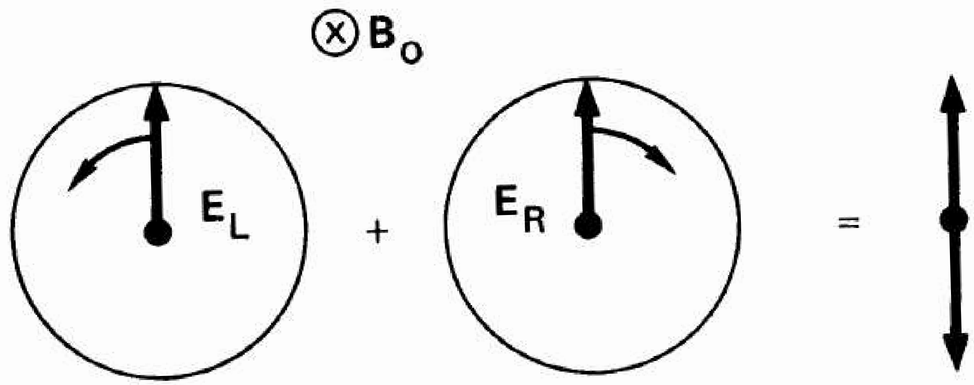


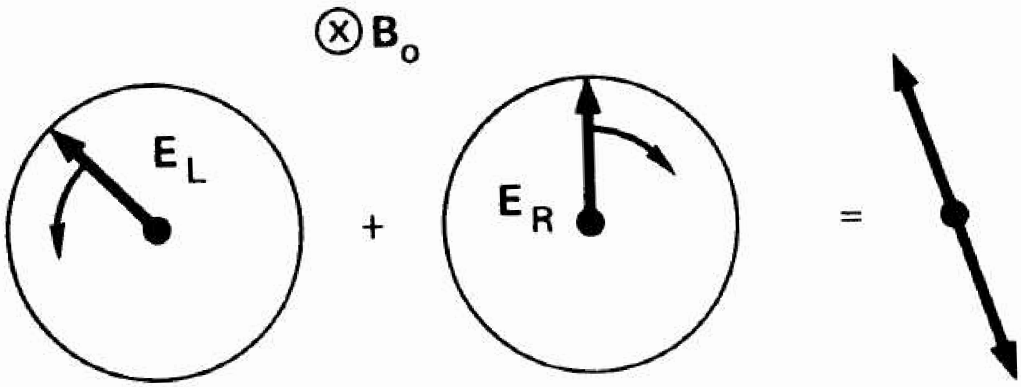
Figure 9: Diagram showing whistlers are created. The channels A, B, and C correspond to the signals in the spectrogram.

Faraday rotation

R waves have higher phase speed than L waves. This means that a plane-polarized wave sent along a magnetic field will suffer a rotation of its plane of polarization. This is because a linear polarized wave can be represented in terms of a linear superposition of two circular polarized waves, which will propagate as L and R waves with different speed, and at the end will have a different phase relation.



A plane-polarized wave as the sum of left- and right-handed circularly polarized waves.



After traversing the plasma, the *L* wave is advanced in phase relative to the *R* wave, and the plane of polarization is rotated.