Plasma waves. Landau damping.

[Chen, p. 240-261]

We have discussed that collisions in plasma may result in energy dissipation. Landau in 1946 found a completely new mechanism which operates even in the absence of collisions. As an example of this mechanism we consider small-amplitude Langmuir waves in the case when thermal motion of particles play an active role.

In lecture 6, we have considered one-dimensional plasma waves using the Vlasov equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0$$

For small deviations f_1 from an equilibrium distribution f_0:

$$f = f_0 + f_1$$

we have:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0,$$

where electric field E is produced by the oscillating electrons:

$$\frac{\partial E}{\partial x} = 4\pi q = -4\pi e \int f_1 dv$$

where q is the oscillatory charge density.

We seek the solution in the form of plane waves:

$$f_1(x, v, t) = \hat{f}_1(v) e^{-i\omega t + ikx}$$

$$-i(\omega - kv)\hat{f}_1 = \frac{e\hat{E}}{m}\frac{\partial f_0}{\partial v}$$

$$ik\hat{E} = -\frac{4\pi e^2 i\hat{E}}{m}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}dv$$

$$ik\hat{E}\underbrace{\left(1 + \frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}dv\right)}_{D(k,\omega)\ -\text{plasma dispersion function}} = 0$$

Equation

$$D(k, \omega) = 0$$

defines the dispersion relation $\omega = \omega(k)$. Sometimes, $D(k, \omega)$ is called "plasma dielectric function", equation for $\hat{E}$ can be written as

$$\nabla \cdot [D(k, \omega)\vec{E}] = \nabla \cdot \vec{D} = 0,$$

where $\vec{D}$ is dielectric displacement.

We have derived the dispersion relation for plasma waves assuming that $\omega > kv$:

$$\frac{1}{\omega - kv} = \frac{1}{\omega} + \frac{kv}{\omega^2} + \frac{k^2v^2}{\omega^3} + \frac{k^3v^3}{\omega^4} + ...$$

Using

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} dv = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v\, dv = -n$$

we obtain:

$$1 - \frac{4\pi e^2 kn}{m\omega^2 k} = 0$$

or

$$\omega_p^2 = \frac{4\pi e^2 n}{m}$$

- plasma frequency.

Next terms of the expansion give:

$$\omega^2 = \omega_p^2 + 3v_T^2 k^2.$$

The integral in the plasma dispersion relation has singularity at $\omega = kv$. However, in reality ω is complex because of some damping processes, and the integral calculated for complex ω is not singular. The damping occurs even without collisions. This effect is called Landau damping.

The physical reason is in the resonant interaction between the waves and particles. Electrons moving with velocities close to the wave phase velocity may absorb wave energy. This process is reverse to the Cherenkov effect when a particle moving with velocity faster than the phase speed of light in a medium emits waves.

Consider first a simple physical description of the wave-particle interaction.

Let

$$E = E_0 \sin(kx - \omega t)$$

is the electric field strength in a plasma wave propagating in the x direction with the phase speed $v_p = \omega/k$. Then the electrostatic potential in this wave can be found from equation:

$$\nabla\phi = -E$$

$$\phi = \frac{eE_0}{k}\cos(kx - \omega t) = \phi_0 \cos(kx - \omega t).$$

The particle with the relative velocity $u = v - v_p$ such as

$$\frac{mu^2}{2} < e\phi_0$$

are trapped by the wave. The maximum relative velocity of the trapped particles is

$$\frac{mu_m^2}{2} = \frac{eE_0}{k} \quad or \quad u_m^2 = \frac{2eE_0}{km}$$

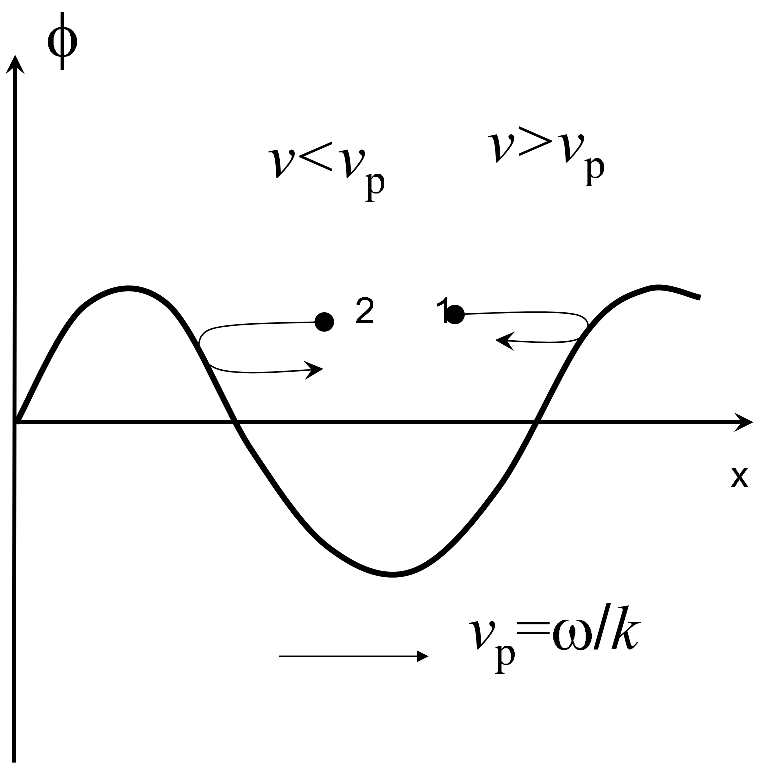

Figure 1: Illustration of the energy exchange between the resonant particles and wave.

Particles with speed $v > v_p$ (type "1") will give their energy to the wave in reflections art the potential well, and particles with $v < v_p$ (type "2") will take energy of the moving wave well. Looking at the distribution function we can conclude that there more particles of type 2 than particles of type 1. Hence the wave will decay.

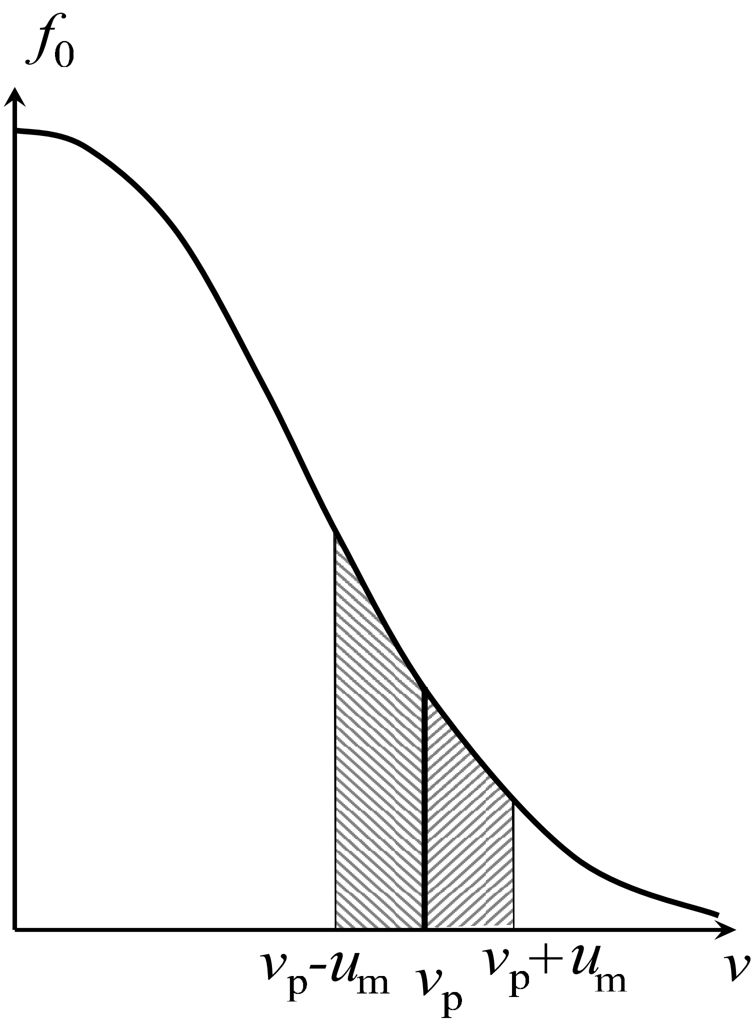

Figure 2: Distribution function f_0 showing particles interacting with the wave.

The energy transfer from particle "1" to wave is

$$\Delta E = \frac{mv^2}{2} - \frac{m(2v_p - v)^2}{2} = 2m(v - v_p)v_p$$

The collision rate can be estimated as

$$\nu_1 \sim \frac{v - v_p}{\lambda}$$

where $\lambda = 2\pi/k$ is the wave length. Then the energy transfer rate for particles with velocity v is:

$$\Delta W = \Delta E \nu_1 = \frac{2m(v - v_p)^2}{\lambda} v_p.$$

Integrating for trapped particles with velocity range $[v_p, v_p + u_m]$ and subtracting a similar integral for particles taking the wave energy, with velocities $[v_p - u_m, v_p]$ we obtain

$$\frac{dW}{dt} = \int_{v_p}^{v_p+u_m} \frac{2mv_p}{\lambda}(v - v_p)^2 f_0(v)dv -$$

$$-\int_{v_p-u_m}^{v_p} \frac{2mv_p}{\lambda}(v - v_p)^2 f_0(v)dv$$

Approximating

$$f_0(v) = f_0(v_p) + \left.\frac{\partial f}{\partial v}\right|_{v_p} (v - v_p)$$

and calculating the integrals we obtain:

$$\frac{dW}{dt} = \frac{2mv_p}{\lambda} \left.\frac{\partial f_0}{\partial v}\right|_{v_p} \frac{u_m^4}{2}$$

$$\frac{dW}{dt} = \frac{2e^2 E_0^2 \omega}{\pi m k^2} \left.\frac{\partial f}{\partial v}\right|_{v=\omega/k}.$$

Since the wave energy is

$$W = \frac{E_0^2}{4\pi}$$

we obtain:

$$\frac{dW}{dt} = W\frac{8e^2\omega}{mk^2} \left.\frac{\partial f}{\partial v}\right|_{\omega/k} = W\frac{2\omega_p^3}{\pi k^2 n} \left.\frac{\partial f}{\partial v}\right|_{\omega/k},$$

where $\omega_p^2 = \frac{4\pi e^2 n}{m}$ is the squared plasma frequency.

Since in Maxwellian plasma $\partial f/\partial v < 0$, the plasma waves decay because of the resonant interaction with electrons. However, if $\partial f/\partial v > 0$ then the wave will gain the energy from electrons. This happens, for instance, for plasma beams (so-called, beam instability).

The decay/growth rate is

$$\gamma = \frac{1}{W}\frac{dW}{dt} = \frac{2\omega_p^3}{\pi k^2 n}\left.\frac{\partial f}{\partial v}\right|_{\omega/k}.$$

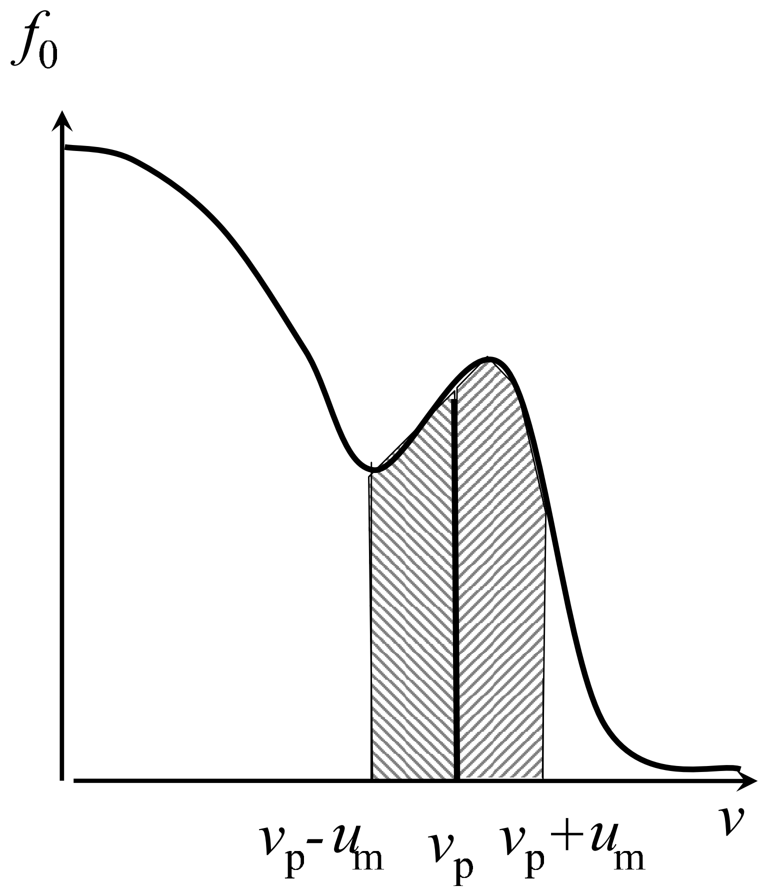

Figure 3: Distribution function f_0 with an electron beam showing particles interacting with the wave.

Landau damping from the Vlasov equation

Consider the plasma dispersion equation obtained from the Vlasov equation:

$$1+\frac{4\pi e^2}{mk}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega-kv}dv=0.$$

Since we know that plasma waves decay, ω is complex, and the integral can be calculated as a contour integral in the complex v plane, using the residue theorem:

$$\int_{C_1}Gdv+\int_{C_2}Gdv=2\pi iR(\omega/k)$$

where G is the integrand, C_1 is the path along the real axis, C_2 is the semicircle at the infinity, and $R(\omega/k)$ is the residue at ω/k.

We consider a simple case when the damping is weak, so that the singular point is close to the real axis. We use the integration contour prescribed by Landau: a straight line along the $Re(v)$ axis with a small semicircle around the pole. In going around the pole, one obtains $2\pi i$ times half the residue there.

Hence,

$$I = \int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial v}}{\omega - kv} dv = P \int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial v}}{\omega - kv} dv +$$

$$+ \int_{\omega/k-\epsilon}^{\omega/k+\epsilon} \frac{dv}{\omega - kv} \left. \frac{\partial f}{\partial v} \right|_{\omega/k}$$

where P stands for the Cauchy principal value.

$$\int_{\omega/k-\epsilon}^{\omega/k+\epsilon} \frac{dv}{\omega - kv} = -\frac{1}{k} \int_{-\epsilon}^{\epsilon} \frac{dz}{z} = -\frac{i}{k} \int_{\pi}^{2\pi} d\phi = -\frac{i\pi}{k}$$

where $z = \epsilon \exp{(i\phi)}$ - complex variable.

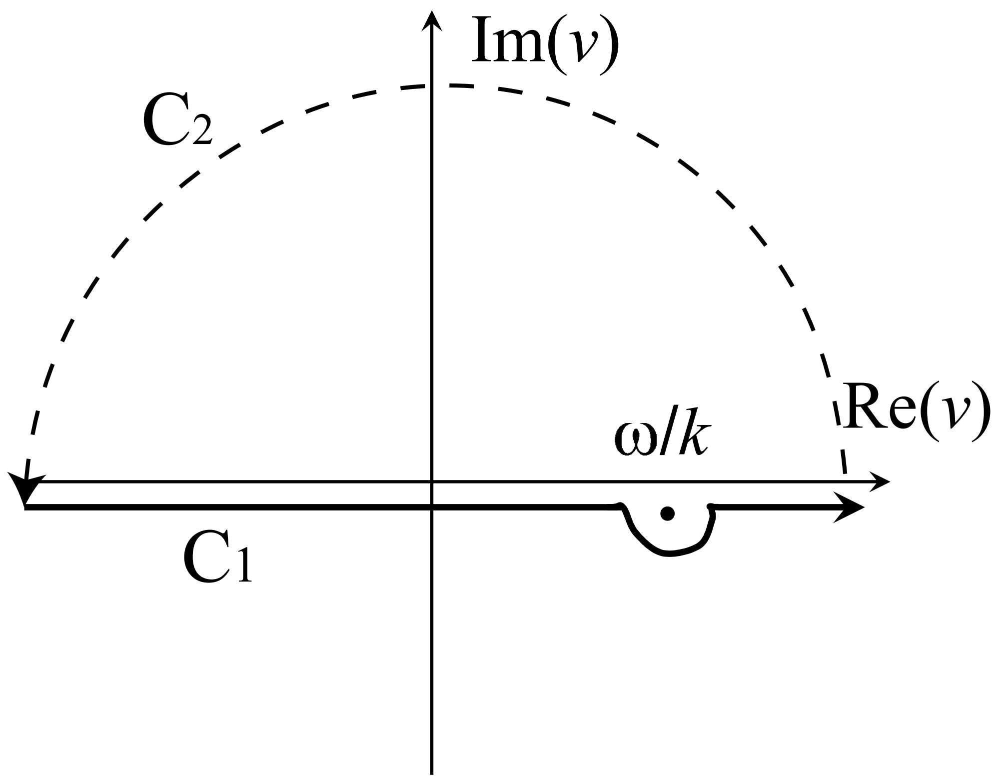

Figure 4: Integration contour in the complex v plane for small $Im(\omega)$.

Thus,

$$I = P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial v}}{\omega - kv} dv - \frac{i\pi}{k} \left.\frac{\partial f}{\partial v}\right|_{\omega/k}$$

Most contribution to the first integral comes from where $\partial f_0/\partial v$ is large, that is where $v \ll \omega/k$.

Expanding

$$\frac{1}{\omega - kv} = \frac{1}{\omega} + \frac{kv}{\omega^2} + \frac{k^2v^2}{\omega^3} + ...$$

we obtain the dispersion relation:

$$1+\frac{4\pi e^2}{mk}\left[\frac{k}{\omega^2}(-n)-\frac{i\pi}{k}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}\right]=0$$

or

$$1-\frac{\omega_p^2}{\omega^2}-\frac{i\pi\omega_p^2}{nk^2}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}=0$$

Since the damping term is small we seek for solution in the form

$$\omega=\omega_p-i\gamma$$

where γ is the decay rate, that is

$$e^{-i\omega t}=e^{-i\omega_p t-\gamma t}.$$

$$1-\frac{\omega_p^2}{(\omega_p-i\gamma)^2}-\frac{i\pi\omega_p^2}{nk^2}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}=0$$

$$1-\frac{\omega_p^2}{\omega_p^2}\left(1+2\frac{i\gamma}{\omega_p}\right)-\frac{i\pi\omega_p^2}{nk^2}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}=0$$

$$-2\frac{i\gamma}{\omega_p} = \frac{i\pi\omega_p^2}{nk^2}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}$$

and, finally,

$$\gamma = \frac{\pi}{2}\frac{\omega_p^3}{k^2 n}\left.\frac{\partial f_0}{\partial v}\right|_{\omega/k}$$

For the Maxwellian distribution this takes form:

$$\gamma = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_p}{(kr_D)^3}e^{-\frac{1}{2(kr_D)^2}}$$

where

$$r_D = \frac{v_T}{\omega_p} = \sqrt{\frac{T}{m\omega_p^2}}$$

is the Debye radius.

Thus, waves with short wavelength, $k \sim 1/r_D$, are strongly damped. For long waves the decay is exponentially small. Because the long waves have frequencies close to the plasma frequency this justifies the approximations that we made in deriving the equation for Landau damping.

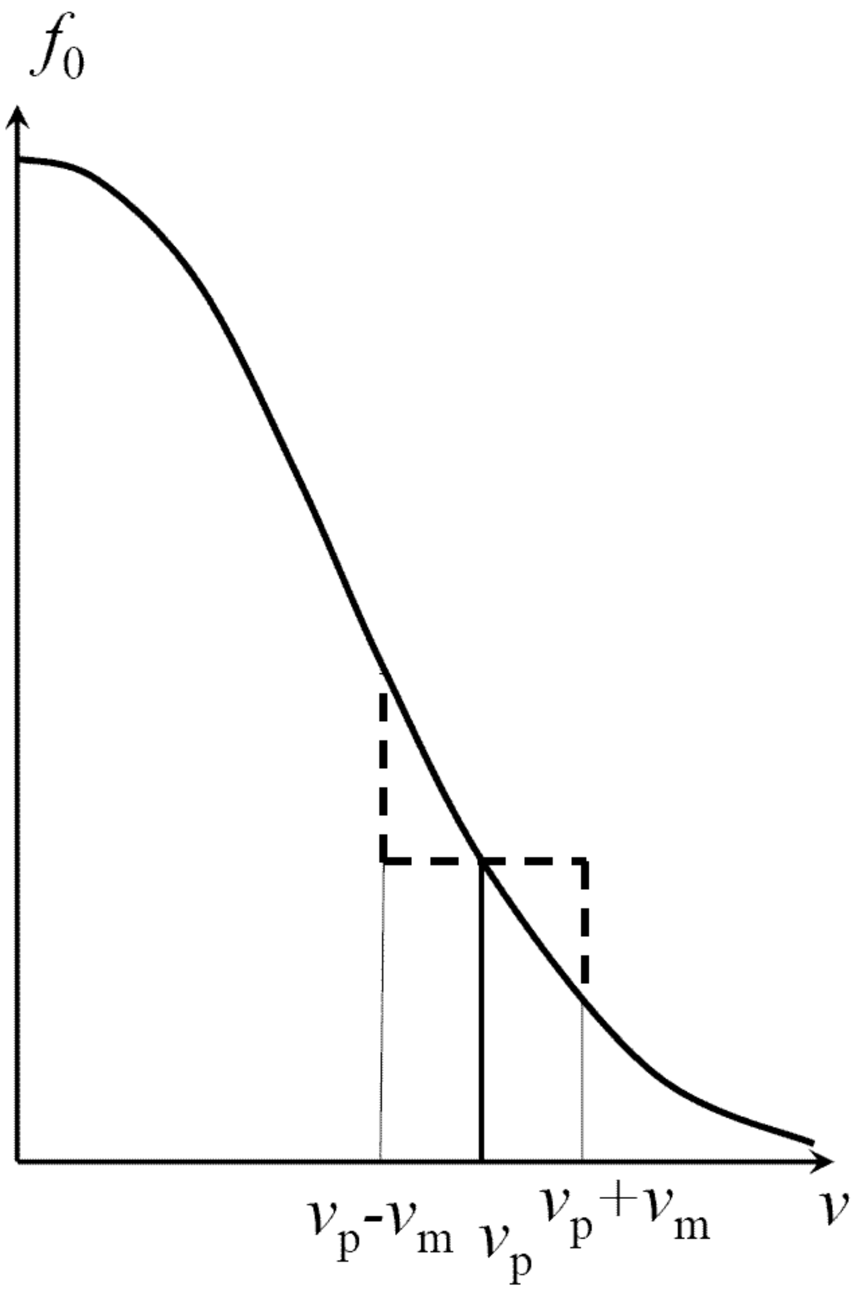

Figure 5: Distortion of a Maxwellian distribution function in the region $v = v_p$ caused by Landau damping.

Beam instability

We can estimate the growth rate of the beam instability when $\partial f/\partial v > 0$. Assume that the beam makes a bump in the distribution function:

$$f \sim \frac{n_1}{\Delta v}$$

where n_1 is the beam density, Δv is the velocity dispersion. Then, the derivative of f can be estimated as

$$\frac{\partial f}{\partial v} \sim \frac{n_1}{\Delta v^2}$$

Hence,

$$\gamma \sim \frac{\omega_p^3}{k^2 n} \frac{\partial f}{\partial v} \sim \omega_p \frac{v^2}{\Delta v^2} \frac{n_1}{n}$$

In non-linear regime, the instability leads to a plateau in the distribution function.

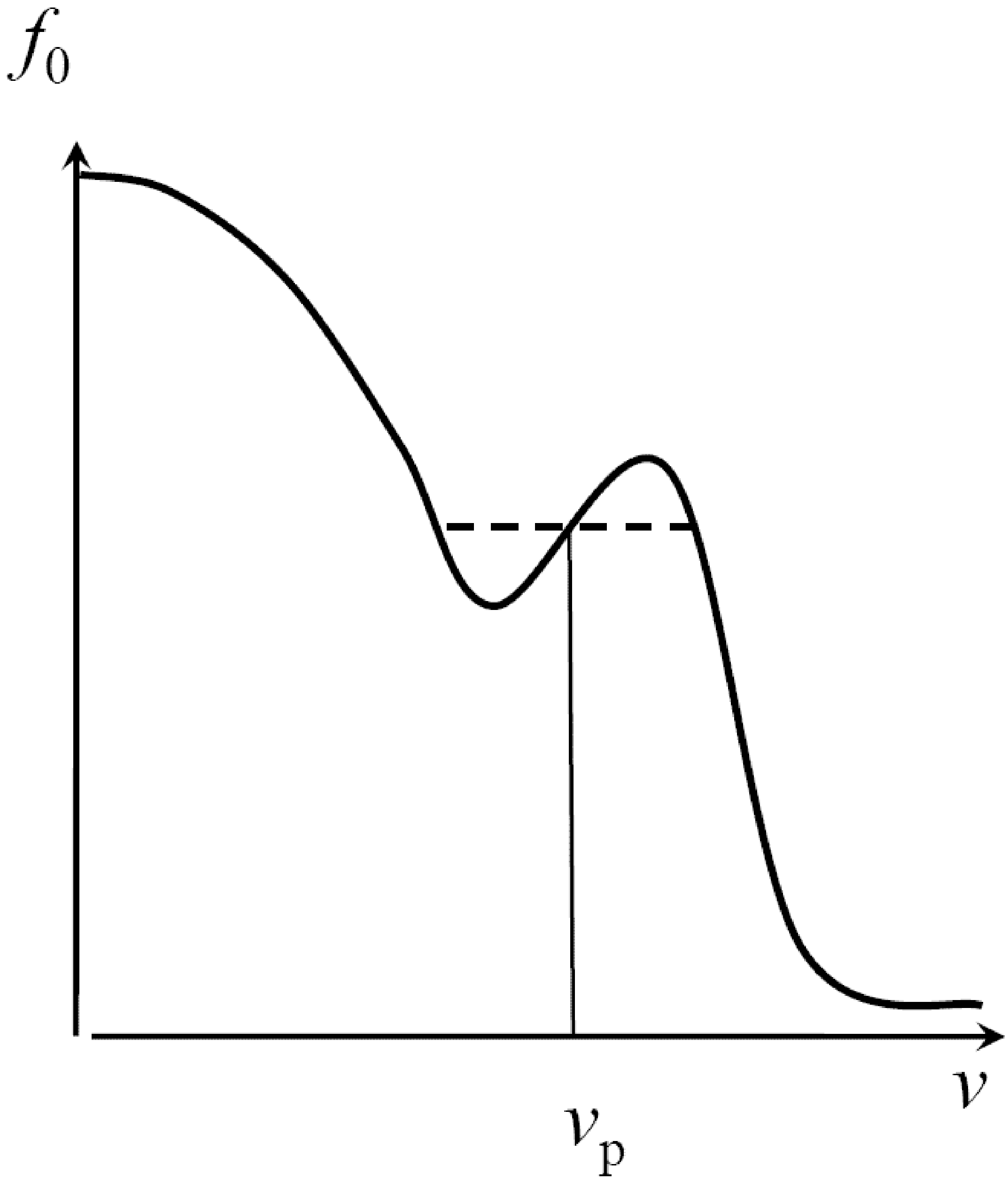

Figure 6: The plateau formation in the distribution function as a result of the beam instability.

Consider the beam instability for monoenergetic beams.

Now we assume that the beam distribution function is very narrow, and that the total distribution function of plasma and beam is

$$f = f_M + f_b$$

where

$$f_b = n_1 \delta(v - v_0)$$

Calculate the plasma dispersion relation:

$$D(k,\omega) = 1 + \frac{4\pi e^2}{mk} \int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial v}}{\omega - kv} dv = 0$$

$$\int_{-\infty}^{\infty} \frac{\frac{\partial f_b}{\partial v}}{\omega - kv} dv = - \int_{-\infty}^{\infty} f_b \frac{\partial}{\partial v} \frac{1}{\omega - kv} dv + \left. \frac{f_b}{\omega - kv} \right|_{-\infty}^{\infty} =$$

$$= -k \int_{-\infty}^{\infty} \frac{f_b}{(\omega - kv)^2} dv = - \frac{kn_1}{(\omega - kv_0)^2}$$

Hence,

$$D(k,\omega) = 1 - \frac{4\pi e^2 n}{n\omega^2} - \frac{4\pi e^2 n_1}{m(\omega - kv_0)^2} =$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - kv_0)^2},$$

where

$$\omega_b^2 = \frac{4\pi e^2 n_1}{m}$$

The dispersion relation is:

$$1 = \frac{\omega_p^2}{\omega^2} + \frac{\omega_b^2}{(\omega - kv)^2} \equiv F(k, \omega)$$

We plot $F(k, \omega)$ vs ω for a fixed k

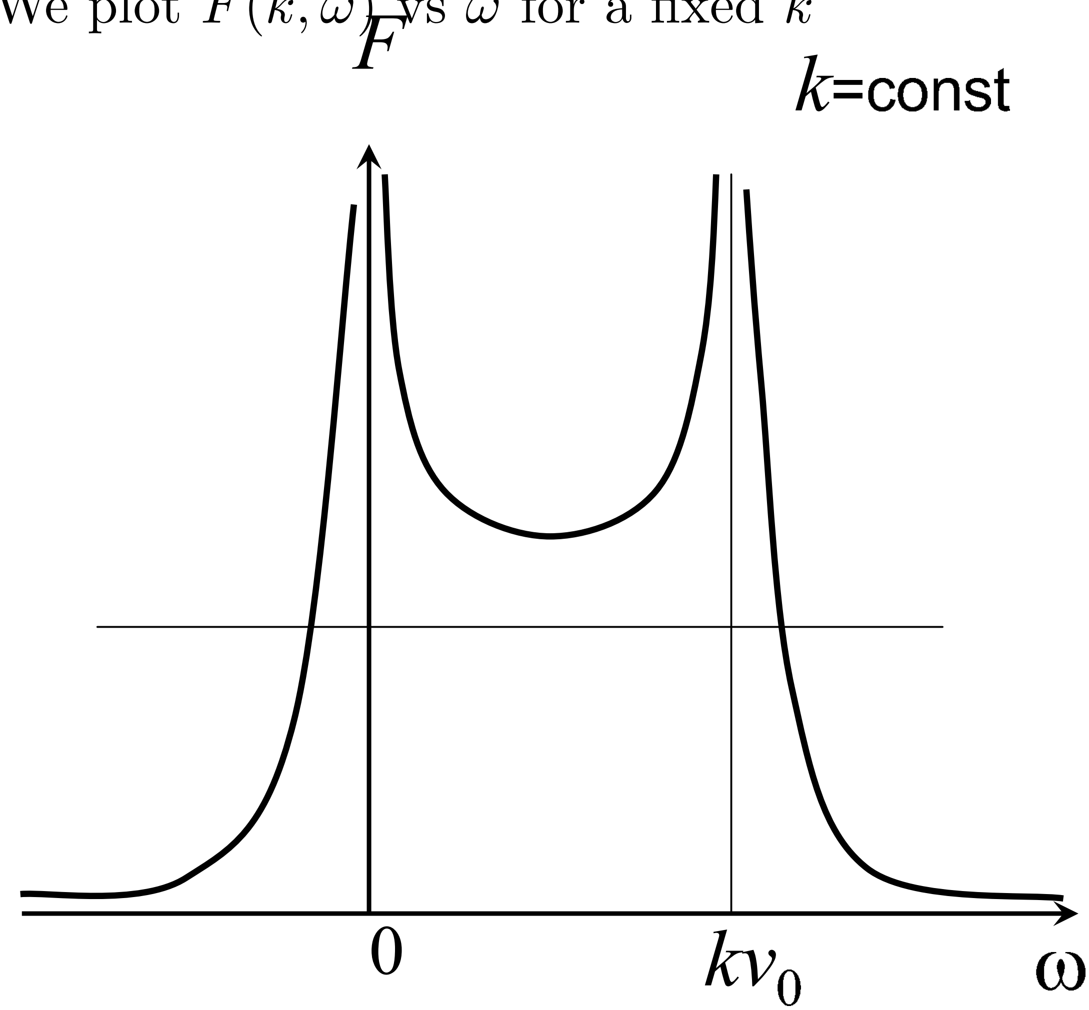

Figure 7: Function $F(k, \omega)$ of the plasma-beam dispersion relation.

It has minimum at:

$$-\frac{2\omega_p^2}{\omega^3} - \frac{2\omega_b^2}{(\omega - kv_0)^3} = 0$$

$$\left(1 - \frac{kv_0}{\omega}\right)^3 + \frac{\omega_b^2}{\omega_p^2} = 0$$

$$\omega = \frac{kv_0}{1 + \left(\frac{\omega_b}{\omega_p}\right)^{2/3}}$$

The minimum value of F is:

$$\min F = \frac{\omega_p^2}{k^2 v_0^2}\left[1 + \left(\frac{\omega_b}{\omega_p}\right)^{2/3}\right]^3$$

When $\min F > 1$ there are complex roots, and hence, instability. This happens when

$$k < \frac{\omega_p}{v_0}\left[1 + \left(\frac{\omega_b}{\omega_p}\right)^{2/3}\right]^{3/2}$$

This means that the long wavelength wave are unstable, excited by the beam.

The maximum growth rate of this instability will occur when

$$kv_0 = \omega_p,$$

the resonance between the oscillations of plasma ($\omega = \omega_p$) and the beam ($\omega = kv_0$). We find the maximum growth rate by seeking the solution of the dispersion relation in the form:

$$\omega = \omega_p - i\gamma :$$

$$1 = \frac{\omega_p^2}{(\omega_p - i\gamma)^2} - \frac{\omega_b^2}{\gamma^2}$$

$$1 = \frac{\omega_p^2}{\omega_p^2}\left(1 + \frac{i\gamma}{\omega_p}\right) - \frac{\omega_b^2}{\gamma^2}$$

$$\frac{\omega_b^2}{\gamma^2} = \frac{i\gamma}{\omega_p^2}$$

$$\gamma^3 = -i\omega_p\omega_b^2$$

or, finally,

$$\gamma \sim \omega_p \left(\frac{\omega_b^2}{\omega_p^2}\right)^{1/3} \sim \omega_p \left(\frac{n_1}{n}\right)^{1/3}$$

Comparing this with the formula for non-monoenergetic beam:

$$\gamma \sim \omega_p \frac{v^2}{\Delta v^2}\frac{n_1}{n}$$

we find that a beam can be considered as monoenergetic when

$$\frac{\Delta v}{v} \ll \left(\frac{n_1}{n}\right)^{1/3} .$$

This instability leads to modulation of the electron beam by the electric field in plasma waves, and in non-linear regime to bunching - trapping of electrons by plasma waves.