Plasma radiation.

([11], Chap.V, p. 246-264)]

Radiation is one of the main channels of energy loss in plasma. The radiation spectrum of an opaque plasma heated a uniform temperature is the black body radiation spectrum described by the Planck function

$$S_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{T}} - 1}.$$

The integrated energy emitted by unit area per unit time is σT^4.

If a partially ionized plasma is transparent in the continuum and emits and absorbs in atomic lines then the radiation spectrum consists of atomic emission lines, where the line radiation for a given frequency is in the thermodynamic equilibrium with the plasma. The hight of the spectral emission lines corresponds to the Planck function. The integrated energy flux in the line spectrum is small. It decreases with the temperature increase.

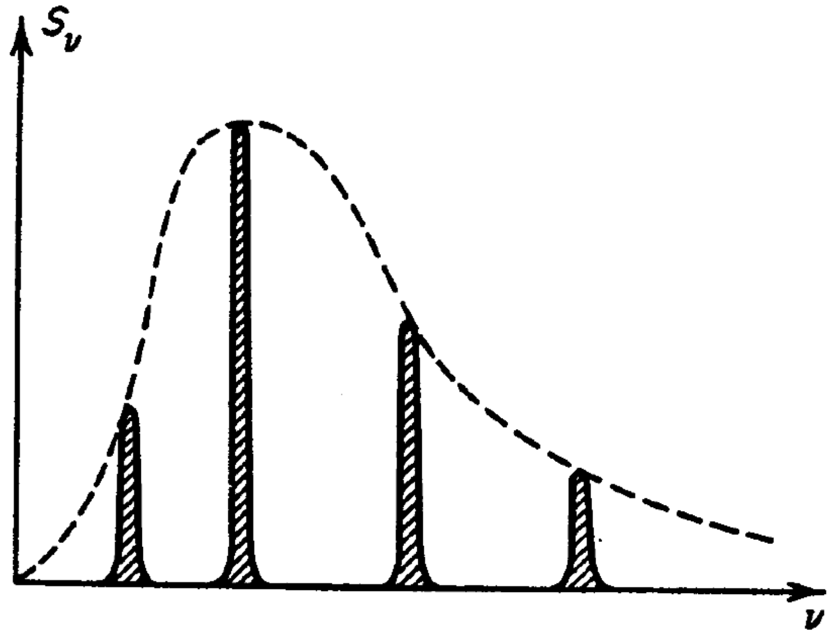

Figure 1: Emission spectrum of a heated body which is transparent in continuum but opaque in atomic lines. The dashed curve is the Planck spectrum.

Fully ionized plasma radiates in continuum. We consider two basic types of continuum radiation

1. bremsstrahlung (free-free transitions - particle scattering)
2. recombination (free-bound transition - electrons are captured in electron-ion collisions)

Consider bremsstrahlung and estimate the radiation intensity in collisions of electrons with ions of charge Ze.

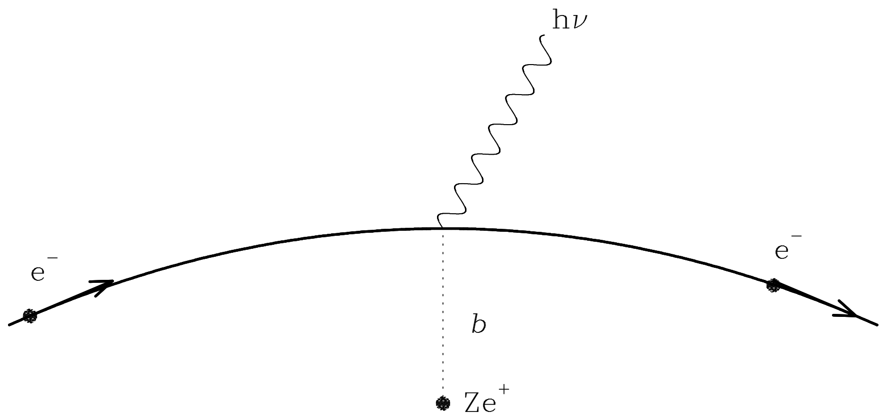

Figure 2: Bremsstrahlung radiation. Electric forces acting on the electron during a collision with an ion cause the emission of photons.

From classical physics we know that accelerated charge particle emit dipole radiation:

$$I = \frac{2e^2\dot{v}^2}{3c^3}$$

(For a simple derivation see: Longair, M. High Energy Astrophysics, Vol.1, Sec.3.2.2, p.62, 1992)

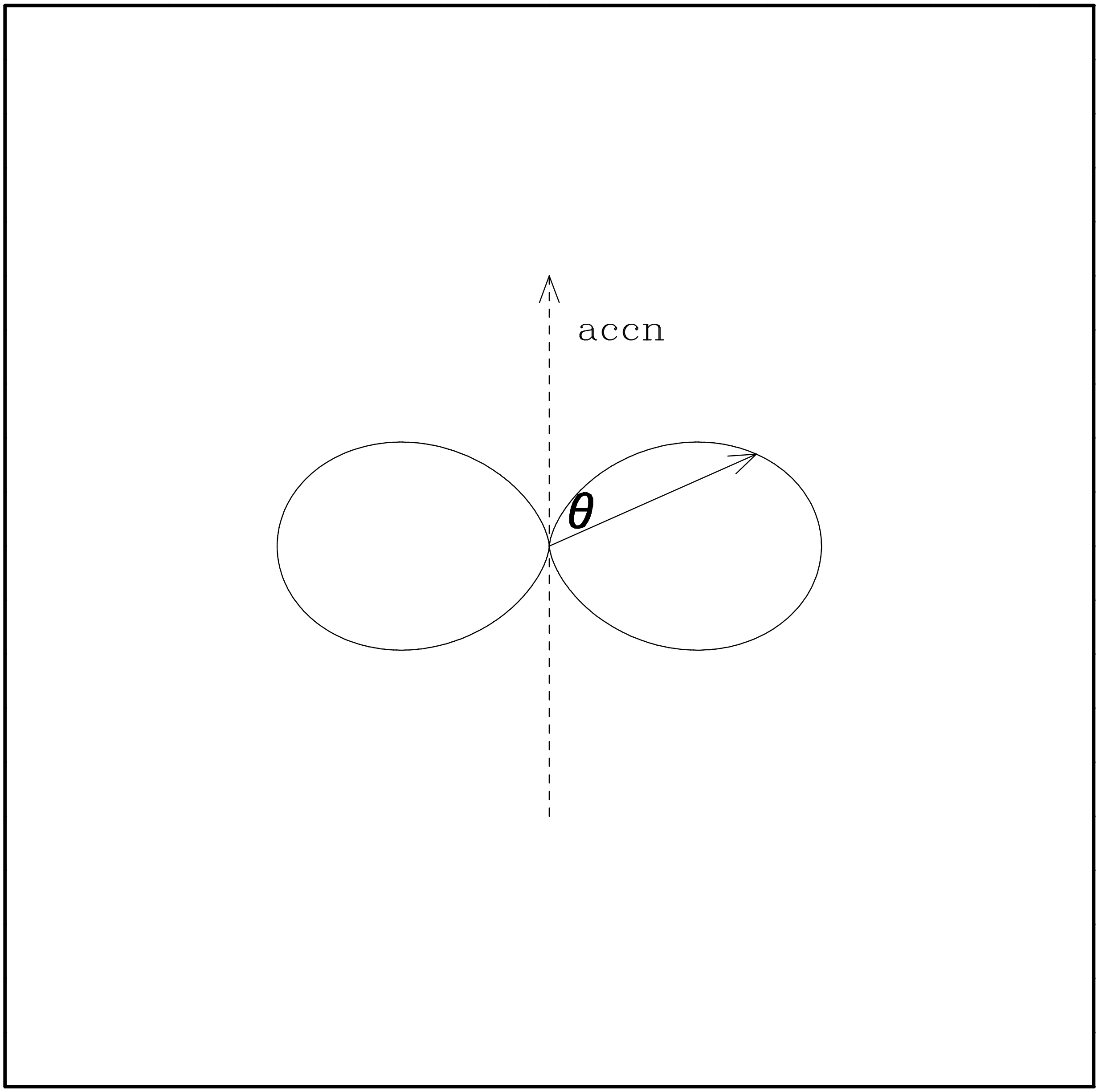

Figure 3: Dipolar emission from an accelerated charge. Maximum emission is perpendicular to the direction of acceleration, and the intensity is proportional to $\sin^2\theta$. The radiation is polarized: the electric field vector is parallel to the acceleration vector.

Since the thermal velocity of ions is much smaller than the thermal velocity of electrons we consider ions as stationary. First, consider collisions for large impact parameter ρ:

$$\rho \gg \frac{2Ze^2}{mv^2}$$

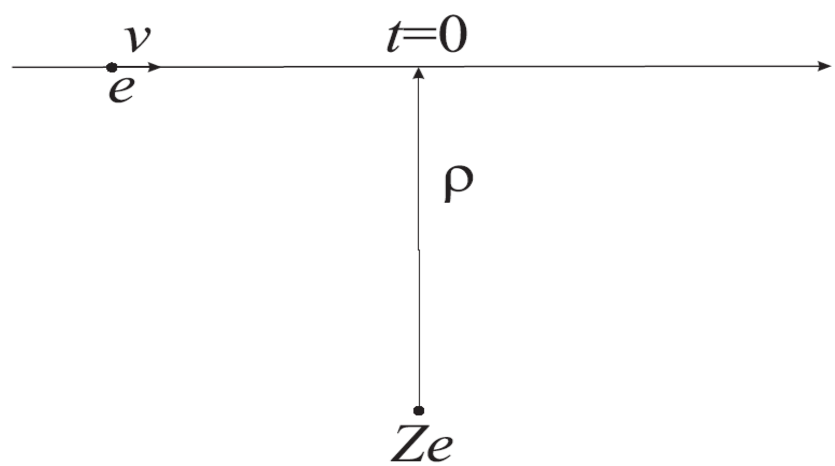

Figure 4: Electron-ion collisions for large impact parameter ρ.

Then acceleration of electron in the electric field of an ion is:

$$\dot{v} = \frac{Ze^2}{mr^2} = \frac{Ze^2}{m(\rho^2 + v^2t^2)}$$

where r is the distance between the particles,

$$r^2 = \rho^2 + v^2 t^2$$

Then the energy emitted by electron in unit time is

$$I = \frac{2e^2 \dot{v}^2}{c^3} = \frac{2}{3} \frac{Z^2 e^6}{m^2 c^3} \frac{1}{r^4}.$$

The total energy emitted by electron

$$\Delta E = \int_{-\infty}^{\infty} I dt = \frac{2Z^2 e^6}{3m^2 c^3} \int_{-\infty}^{\infty} \frac{dt}{(\rho^2 + v^2 t^2)^2} =$$

$$= \frac{2}{3} \frac{Z^2 e^6}{m^2 c^3 v \rho^3} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}}_{\pi/2} = \frac{\pi Z^2 e^6}{3m^2 c^3 v \rho^3}$$

The radiation power for an uniform flux of electrons with density n_e and velocity v:

$$P_1 = \int n_e v \Delta E 2\pi \rho d\rho = \frac{2\pi^2 Z^2 e^6 n_e}{3m^2 c^3} \int \frac{d\rho}{\rho^2}$$

If we integrate for all ρ from 0 to infinity then the integral is diverging at 0. The reason is not just the large ρ approximation. It is even more diverging for close collisions. The lower limit is determined from quantum mechanics. The uncertainty principle

$$\Delta r \Delta p \sim \hbar$$

prohibits electrons from getting closer than

$$\rho_{\rm min} \sim \frac{\hbar}{mv} \sim \lambda_B$$

where λ_B is the De Broglie wavelength.

Hence,

$$P_1 \simeq \frac{2\pi^2}{3} \frac{Z^2 e^6 n_e}{m^2 c^3} \frac{1}{\lambda_B}$$

We have to check when the condition of large impact distances is compatible with the lower limit that is

$$\rho_{\rm min} > \frac{2Ze^2}{mv^2}$$

this means that

$$\frac{\hbar}{mv} > \frac{2Ze^2}{mv^2}$$

or

$$\frac{\hbar v}{2Ze^2} > 1$$

hence, the electron energy

$$\frac{mv^2}{2} > \frac{me^4 Z^2}{2\hbar^2} = 13.6 eV \cdot Z^2$$

Thus, in this case the electron energy is higher than the ionization energy.

The total radiation power from a unit volume of plasma is:

$$P = P_1 n_i \sim \frac{Z^2 e^2 v}{m^2 c^3 \hbar} n_e n_i$$

The radiation frequency can be estimated from the following. For a given impact parameter ρ the duration of the acceleration impulse is

$$t \sim \frac{\rho}{v}$$

Hence, the characteristic frequency is

$$\omega \sim 1/t \sim \frac{v}{\rho}$$

Using the minimal distance for ρ estimate we get

$$\omega \sim \frac{mv^2}{\hbar}$$

or for the photon energy

$$\hbar\omega \sim mv^2 \sim 2T_e$$

The precise theory for a Maxwellian distribution gives the same answer: the maximum radiation frequency is emitted by photons with energy $\sim 2T_e$. The radiation spectrum is flat with a cut-off frequency at $\hbar\omega \sim 2T_e$.

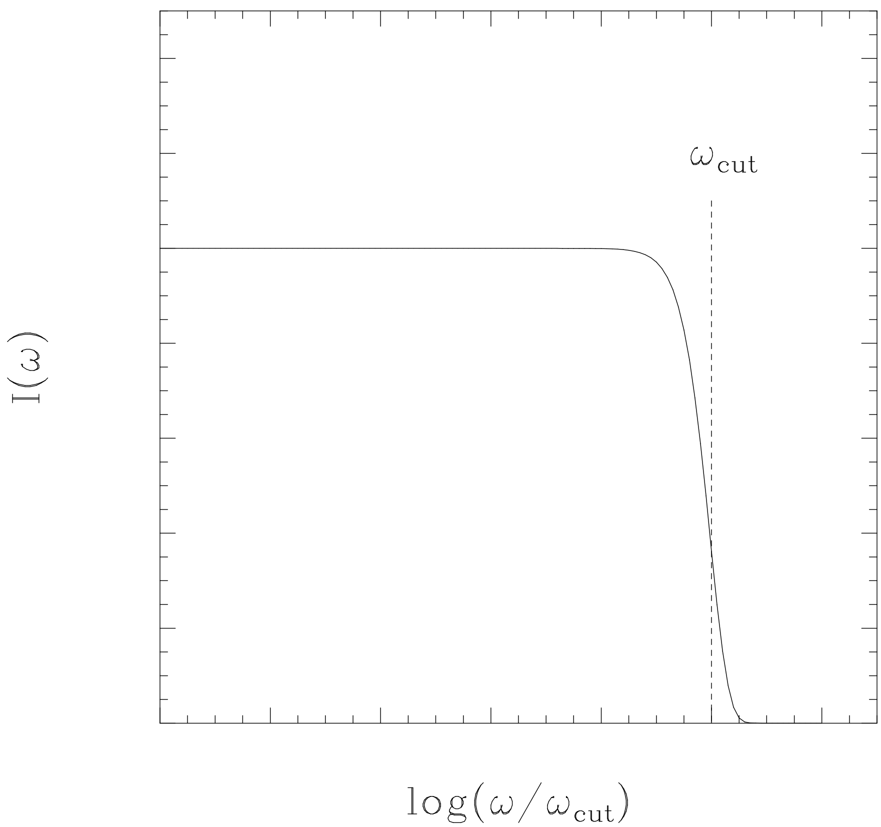

Figure 5: Spectrum of bremsstrahlung radiation. The spectrum is flat up to the cut-off frequency ω_{cut}, corresponding to photons with the thermal energy: $\hbar\omega = 2T_e$.

The averaged over the velocity distribution radiation power corresponds to $v \sim v_{T_e} \sim \sqrt{T_e/m}$

with some coefficient:

$$P_{\mathrm{brem}} = A\frac{Z^2 e^6 n_e n_i {T_e}^{1/2}}{m^{3/2} c^3 \hbar}$$

where $A \approx 7.7$.

Practical formula:

$$P_{\mathrm{brem}} = 1.69 \cdot 10^{-25} Z^2 n_e n_i \sqrt{T_{\mathrm{eV}}} \quad \frac{\mathrm{erg}}{\mathrm{cm}^2\mathrm{s}}$$

Since electron looses almost the whole kinetic energy to radiation in the collisions it may be captured by the ion. However, for electrons with energies higher than the ionization energy this probability is small.

The energy levels in a hydrogen-like atom are

$$E_n = -\frac{I_H Z^2}{n^2}$$

where

$$I_H = \frac{me^4}{2\hbar^2} = Ry$$

Then the photon energy radiated by electron captured to quantum state n is

$$\hbar\omega = \frac{mv^2}{2} + \frac{I_H Z^2}{n^2}$$

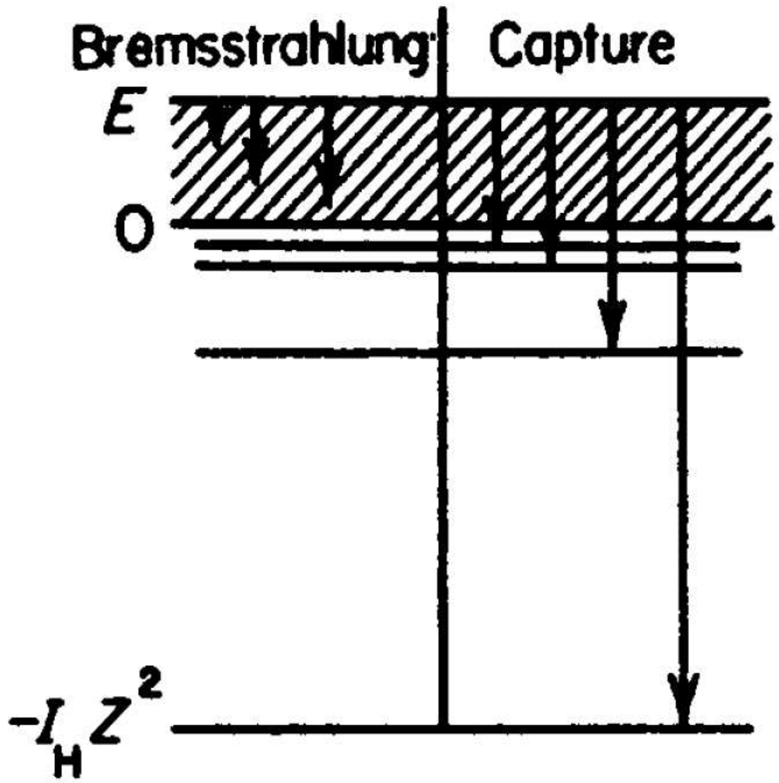

Figure 6: Diagram showing the relationship between the energy interval for bremsstrahlung and recombination radiation.

Consider radiation in a finite frequency range $\Delta\omega$. This corresponds to capture into Δn number of energy states:

$$\hbar\omega \sim \frac{2I_H Z^2}{n^3}\Delta n$$

Since the radiation frequency can be estimated as

$$\omega \sim v/\rho$$

we can find the range of the impact parameter corresponding to the capture into Δn states:

$$\frac{\hbar v}{\rho}\frac{\Delta\rho}{\rho} \sim \frac{2I_H Z^2}{n^3}\Delta n$$

$$\frac{\Delta\rho}{\rho} \sim \frac{2I_H Z^2}{\hbar\omega}\frac{\Delta n}{n^3}$$

Now we calculate the radiation power for the free-bound transition, so-called, **recombination radiation** similarly to bremsstrahlung, but doing summation over possible values of the impact parameter instead of integration:

$$P_r = \sum n_e v \Delta E 2\pi\rho\Delta\rho = \sum \frac{2\pi^2 Z^2 e^6 n_e}{3m^2c^3\rho}\frac{\Delta\rho}{\rho}$$

This can be estimated as

$$P_r \sim \sum P_1 \frac{\Delta\rho}{\rho} \sim \sum P_1 \frac{2I_H Z^2}{\hbar\omega}\frac{\Delta n}{n^3}$$

Obviously, most contribution comes from small

$n \sim 1$. Thus

$$P_r \sim P_1 \frac{2I_H Z^2}{E}$$

where $E = \hbar\omega$ the photon energy. Thus, the power of the recombination radiation is $2I_H Z^2/E$ smaller than the bremsstrahlung radiation:

$$\frac{P_{rec}}{P_{brem}} \sim \frac{2I_H Z^2}{E}$$

here $2I_H \approx 2 \cdot 13.65 \approx 27$. The precise coefficient is 29.

The practical formula for the recombination radiation is

$$P_{\rm rec} = 5 \cdot 10^{-24} Z^4 n_e n_i \frac{1}{\sqrt{T_{\rm eV}}} \quad \frac{\rm erg}{\rm cm^2 s}$$

A small amount of heavy elements may significantly increase the radiation energy losses in plasma because of the Z^4 dependence.

Consider now close collisions. In this case the kinetic energy is smaller than the ionization energy:

$$\frac{mv^2}{2} < \frac{me^4 Z^2}{2\hbar^2}$$

The electron trajectory is not straight. Let r_* be the shortest distance and v_* be the velocity at this distance.

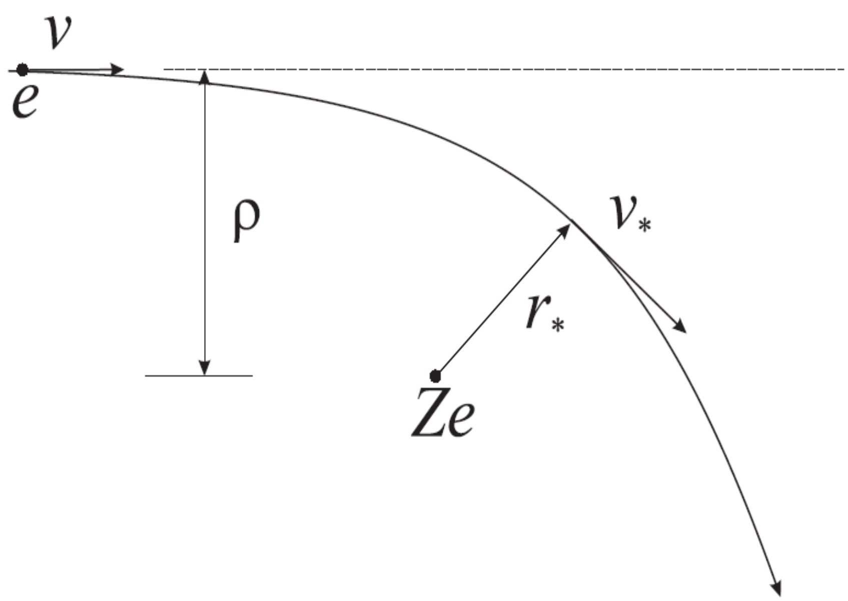

Figure 7: Electron-ion collisions for small impact parameter ρ.

Then the Larmor formula gives

$$I \sim \frac{Z^2 e6}{m^2 c^3 r_*^4}$$

The characteristic duration of the collision is:

$$\Delta t \sim \frac{r_*}{v_*}$$

Hence the total radiation power

$$\Delta E \sim I\Delta t \sim \frac{Z^2 e^6}{m^2 c^3 v_* r_*^3}$$

We determine r_* and v_* from the energy and mass conservation:

$$\frac{mv^2}{2} = \frac{mv_*^2}{2} - \frac{Ze^2}{r_*}$$

$$mv\rho = mv_* r_*$$

If the initial velocity is much smaller the maximum velocity $v \ll v_*$ then

$$v_* \sim \sqrt{\frac{2Ze^2}{mr_*}}$$

$$mv\rho \sim m\sqrt{\frac{2Ze^2}{m}}\sqrt{r_*}$$

Thus

$$r_* \sim \rho \left(\frac{2Ze^2}{\rho m v^2}\right)^{-1}$$

In our case of the small impact parameter

$$\frac{2Ze^2}{mv^2\rho} \gg 1$$

thus $R_* \ll \rho$.

Finally, for a single collision we obtain

$$\Delta E \sim \frac{Z^2 e^6}{m^2 c^3 v \rho^3} \left(\frac{2Ze^2}{mv^2 \rho} \right)^2$$

For an electron flux of density n_e and single ion we get;

$$P_1 = \int n_e v \Delta E 2\pi \rho d\rho \sim \frac{Z^2 e^6 n_e v}{m^2 c^3 v} \int \left(\frac{2Ze^2}{mv^2 \rho} \right)^2 \frac{d\rho}{\rho^2} \sim$$

$$\sim \frac{Z^4 e^{10} n_e}{m^4 c^3 v^4} \int \frac{d\rho}{\rho^4} \sim \frac{Z^4 e^{10} n_e}{m^4 c^3 v^4} \frac{1}{\lambda_B^3} \sim$$

$$\sim \frac{Z^4 e^{10} n_e}{m^4 c^3 v^5} \frac{m^3 v^3}{\hbar^3} \sim \frac{Z^4 e^{10} n_e}{m c^3 \hbar^3 v}.$$

For a Maxwellian distribution $v \sim v_{T_e}$ the radiation from a unit volume is:

$$P_{\text{brem}} \sim \frac{Z^4 e^{10} n_e n_i}{m c^3 \hbar^3 v_{T_e}}$$

We note that this formula is identical to the formula for the recombination radiation. Thus, at low temperature when the electron energy is smaller than the ionization energy the recombination radiation is dominant in the continuum.

When we calculate radiation from a unit volume we have to consider absorption of emitted radiation, which is the inverse process.

Let us calculate the characteristic absorption length of an electromagnetic wave in plasma. Consider electromagnetic wave of amplitude E and frequency ω. Electrons oscillate in this wave with velocity

$$v = \frac{eE}{m\omega}$$

The kinetic energy of electrons in unit volume is

$$n_e \frac{mv^2}{2} = \frac{n_e e^2 E^2}{2m\omega^2} = \frac{E^2}{8\pi} \frac{\omega_p^2}{\omega^2}$$

Because of collisions with ions this energy is converted to heat. Then we write the equation for the energy flux $\mathrm{div}\vec{F} = Q$ where $F = cE^2/8\pi$:

$$\frac{d}{dx}\left(c\frac{E^2}{8\pi}\right) = -\nu_{ei}\frac{\omega_p^2}{\omega^2}\frac{E^2}{8\pi}$$

We find solution

$$\frac{E^2}{8\pi} = \frac{E_0^2}{8\pi} e^{-x/l}$$

where

$$l = \frac{c}{\nu_{ei}} \frac{\omega^2}{\omega_p^2}$$

is the mean free path for photons in plasma - optical depth.

For the collision frequency we have

$$\nu_{ei} = n_i v \sigma_{\rm tr}$$

where the transport cross-section is

$$\sigma_{\rm tr} = \frac{4\pi Z^2 e^4 \Lambda_C}{m^2 v^4}$$

Thus,

$$l \sim \frac{c m^2 v_{T_e}^4 T_e^2 m}{n_i V_{T_e} e^4 Z^2 \hbar^2 e^2 n_e} \sim \frac{c m^{3/2} T_e^{7/2}}{n_e n_i Z^2 e^6 \hbar^2}$$

The corresponding practical formula is

$$l = 2.5 \cdot 10^{37} \frac{T_{\rm eV}^{7/2}}{n_e n_i Z^2} \quad {\rm cm}$$

where n_e and n_i are in cm^{-3}.

Examples.

Laboratory plasma: for $T_e = 10$ eV, $n_i = n_e = 10^{15}$ cm^{-3} we estimate $l = 8 \cdot 10^{10}$ cm.

Solar photosphere: $T_e = 1$ eV, $n_i = 10^{17}$ cm^{-3}, $n_e = 10^{13}$ cm^{-3} (partially ionized plasma): $l \sim 250$ km.

Comparing with the characteristic size of laboratory (~ 1 m) and solar (700,000 km) We conclude that the laboratory plasma is transparent in continuum, but the solar plasma is opaque.

We can estimate the radiative losses of a plasma volume. If L is a characteristic size of the plasma volume then the radiation power S from a unit surface is:

$$S \sim \frac{PL^3}{L^2} \sim PL.$$

If $L \gg l$ then plasma looses energy only from a thin surface layer of thickness l:

$$S \sim Pl.$$

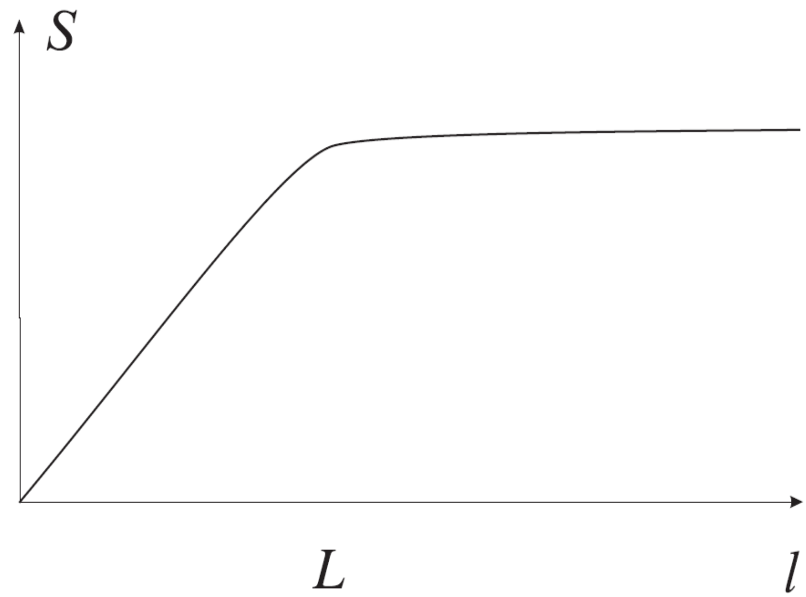

Figure 8: Radiation power of a plasma volume as a function of optical depth l.

Substituting P and l for bremsstrahlung:

$$S \sim \frac{Z^2 e^6 n_e n_i T_e^{1/2}}{m^{3/2} c^3 \hbar} \cdot \frac{c m^{3/2} T_e^{7/2}}{n_e n_i Z^2 e^6 \hbar^2} \sim \frac{T_e^4}{c^2 \hbar^3}$$

This is Stefan-Boltzmann black-body radiation law:

$$S = \frac{\pi^2}{60} \frac{T^4}{c^2 \hbar^3}$$

It can be obtained also by integrating over frequencies the Planck formula

$$S_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{T}}-1}.$$

As we discussed in reality there is also radiation in spectral line. However, the line radiation has only small amount of energy. Plasma transparent in continuum radiates a line spectrum.

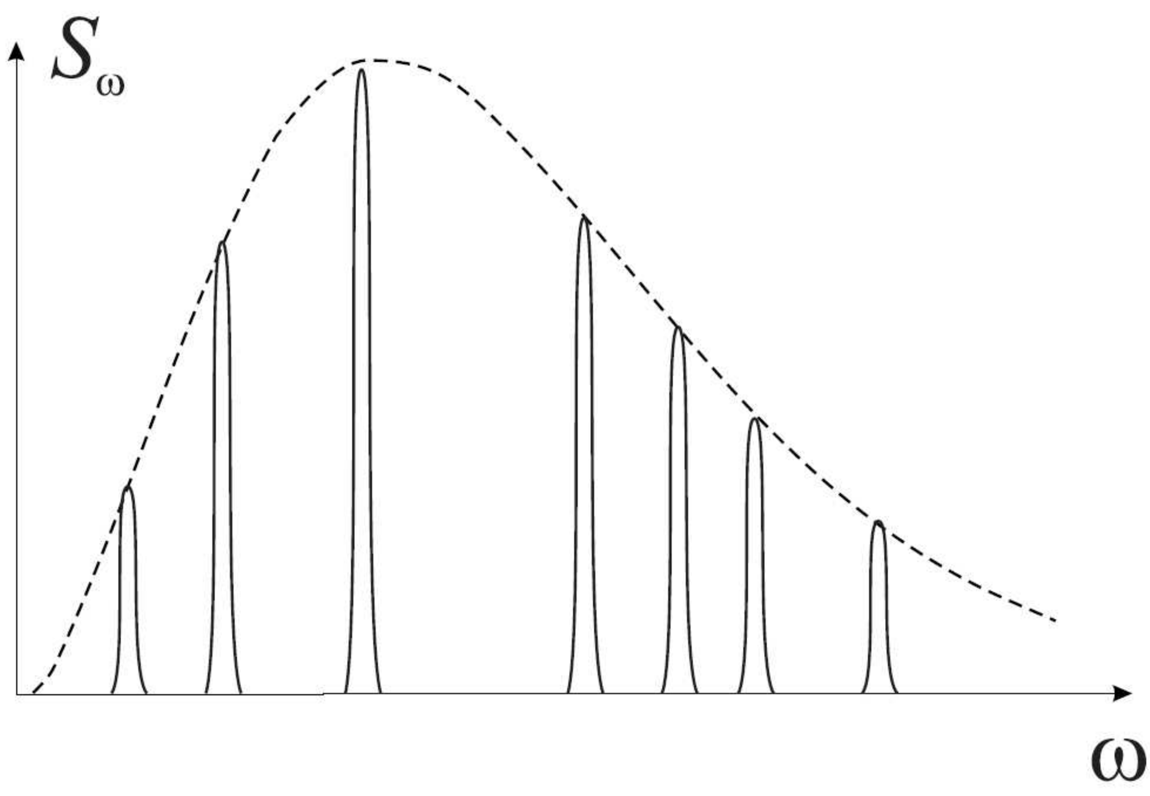

Figure 9: Radiation spectrum of a plasma transparent in continuum, but opaque in spectral lines. The dashed curve shows the Planck spectrum.

Contrary an opaque plasma radiates in continuum, according to the Planck formula, with absorption lines which are formed because the line radiation is formed in a shallow surface layer (due to a small l) where the temperature is lower compared to the temperature of the deeper layer where the continuum radiation is formed.

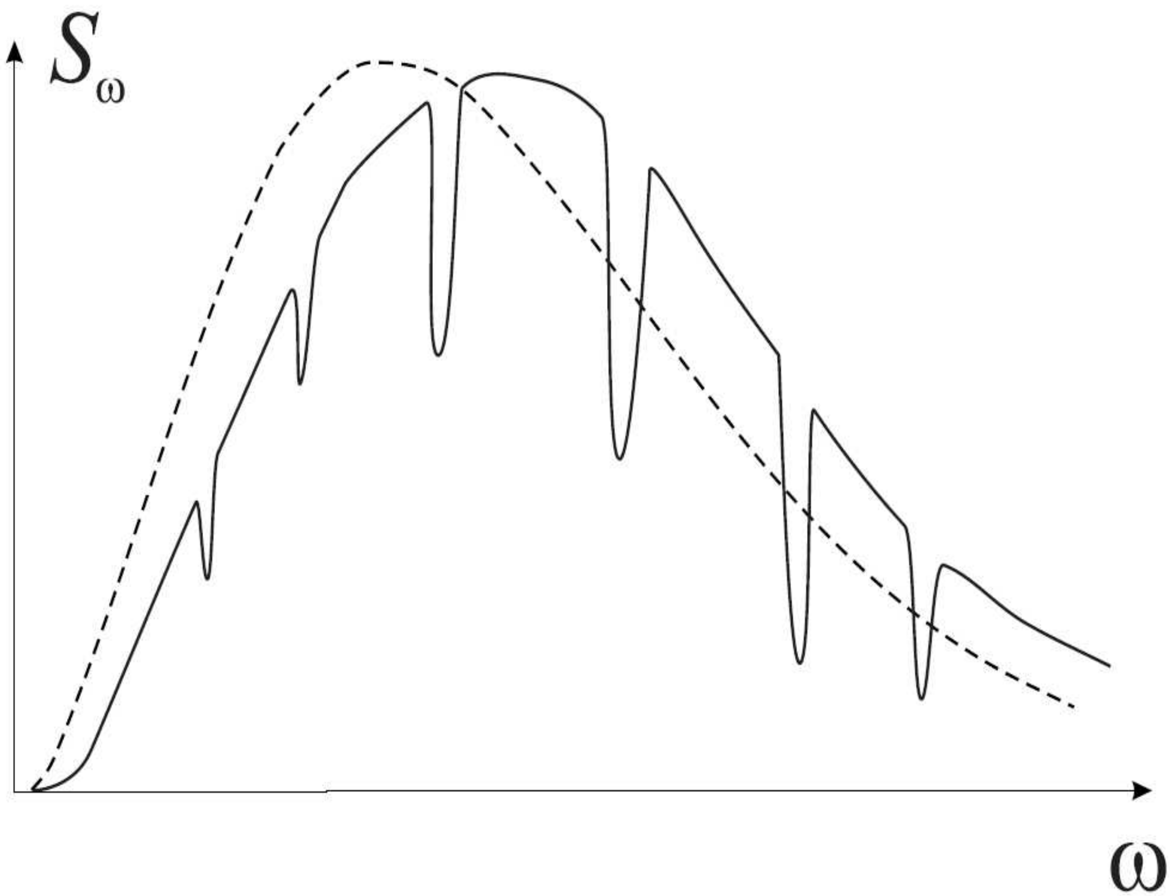

Figure 10: Radiation spectrum of an opaque plasma with temperature decreasing towards the surface. Radiation at lower frequencies is absorbed more than the radiation at higher frequencies. The dashed curve shows the Planck function corresponding to a mean effective temperature of the plasma. The spectrum shows absorption lines. The radiation power in the line centers corresponds to the Planck function for the surface temperature.

In the presence of magnetic field, there is also **cyclotron radiation**. Electrons rotating on circular orbits produce radiation at harmonics of the cyclotron frequency:

$$\omega = l\omega_c = l\frac{eB}{mc}$$

where $l = 1, 2, ...$ Using the Larmor formula for radiation of an accelerating electron we can calculate the total power of cyclotron radiation in a unit volume:

$$P_{cycl} = \frac{2e^2}{3c^3}(\omega_c v_\perp)^2 n_e = \frac{2e^4 B^2 T_e n_e}{3m_e c^5}$$

where $v_\perp$ is the component of electron velocity perpendicular to the magnetic field, $\omega_c v_\perp$ is the centrifugal acceleration of electrons.

Practical formula for cyclotron frequency:

$$\omega_c = 1.8 \cdot 10^7 B_{\text{Gauss}} \quad [sec^{-1}]$$

For $B = 50$ kG, $\omega_c = 9 \cdot 10^{11}$ s^{-1}, and the wavelength $\lambda = 2\pi c/\omega_c \approx 2$ cm.