Dynamo theory

([8], p. 178-196)

Geological data show that the Earth's magnetic field existed for at least the past 3×10^9 years. However, in the absence of external sources of electric currents, magnetic field decays on the time scale:

$$\tau = \frac{4\pi\sigma L^2}{c^2},$$

where conductivity $\sigma \sim 1.5 \times 10^5\ \mathrm{s}^{-1}$, $L \sim 3.5 \times 10^8$cm is the Earth's radius. Then, the decay time:

$$\tau \sim 6 \times 10^{12}\ \mathrm{s}\ \sim 2 \times 10^5\ \text{years}$$

Clearly, some process inside the Earth must maintain the magnetic field. This process of self-generation of magnetic field is called *dynamo*. Similar estimates can be made for stars and galaxies.

The magnetic energy lost via ohmic heating is replenished by the work against the Lorenz force: $\vec{V} \cdot (\vec{j} \times \vec{B})/c$, where $\vec{V}$ is a flow field driven, for instance, by thermal convection. If the flow is sufficiently strong the energy input can overcome the losses.

Consider *kinematic dynamo* theory when the flow field is prescribed, and magnetic field does not affect the flow field.

The basic MHD equation is:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + \frac{c^2 \eta}{4\pi} \nabla^2 \vec{B},$$

where $\eta = 1/\sigma$ is electrical resistivity.

If $\vec{V}$ is prescribed than the equations are linear, and the problem is reduced to find a growing solution, which is linear eigenvalue problem. What type of motion is capable of self-generating a magnetic field?

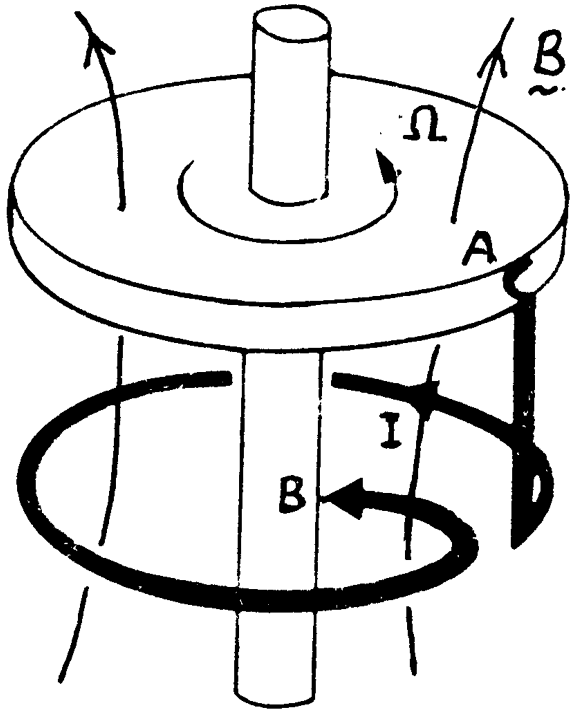

Figure 1: The homopolar disk dynamo.

The homopolar generator

The dynamo process can be illustrated by a simple homopolar disk dynamo. The device consists of a conducting disk which rotates about about its axis by an external force. A twisted wire has a sliding contact with the disk and is connected with the axis. It carries current $I(t)$.

Let's find if this current can grow.

The magnetic field associated with this current has a flux $\Phi = MI$ across the disk, where M is the mutual inductance between the wire and the rim of the disk. The rotation of the disk in the presence of this flux generates electromotive force

$$\mathcal{E} = \frac{d\Phi}{dt} = \frac{\Omega}{2\pi}\Phi = \frac{\Omega}{2\pi}MI$$

The equation for I is written:

$$L\frac{dI}{dt} + RI = \frac{M}{2\pi}\Omega I,$$

where R is the total resistance, and L is the self-inductance.

Solution is:

$$I(t) = I_0 e^{\gamma t},$$

where

$$\gamma = \frac{1}{L}\left(\frac{M}{2\pi}\Omega - R\right)$$

The solution is growing when $\gamma > 0$, or when

$$\Omega > \frac{2\pi R}{M}.$$

Thus, rapid rotation is essential.

Slow and fast dynamos

Let's search for solutions of the MHD equation for a prescribed velocity field $\vec{V}(\vec{r})$. Requirements:

1. solution must be self-contained, that is maintained by motions rather than by currents at infinity; thus $V, B \to 0$ as $r \to \infty$
2. solution must be exponentially growing, that is: $B \propto e^{\gamma t}$, $\gamma > 0$.

In astrophysics, the resistivity is very small, $\eta \to 0$. We can expect that

$$\lim_{\eta \to 0} \gamma \propto \eta^{\alpha}.$$

There can be two types of dynamo (Vainstein & Zeldovich, 1978):

1. $\alpha > 0$ - "slow" dynamo
2. $\alpha = 0$ - "fast" dynamo (the growth rate does not depend on plasma resistivity)

Slow dynamo cannot operate in perfectly conducting fluid (e.g. homopolar generator is a slow dynamo). However, fast dynamo can operate when $\eta = 0$.

When $\eta = 0$ magnetic field lines are frozen into plasma. Thus, stretching fluid will amplify magnetic field.

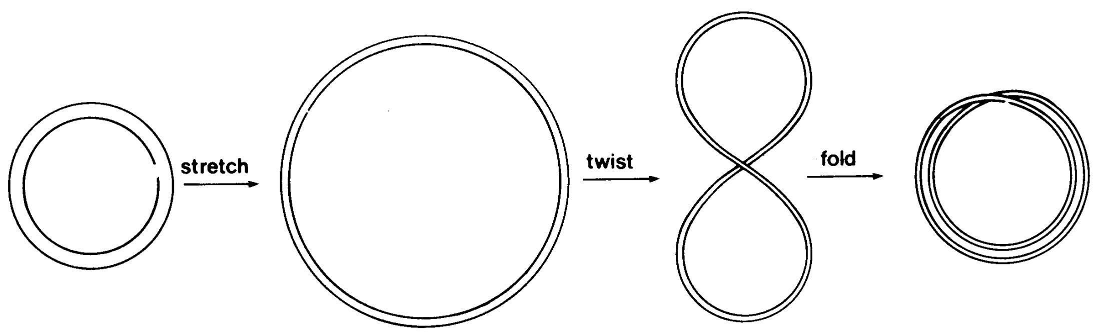

Figure 2: Example of fast dynamo action: a stretch-twist-fold cycle for magnetic field lines (Vainstein and Zeldovich, 1978)

A magnetic fluxtube can be doubled in intensity by taking it around a stretch-twist-fold cycle. The doubling time for this process clearly does not depend on the resistivity: in this sense, this dynamo is a fast dynamo. However, under repeated application of this cycle the magnetic field develops increasingly fine-scale structure. In fact, both velocity and magnetic fields eventually become chaotic.

Fast dynamo is not fully established but needed in astrophysics.

Cowling anti-dynamo theorem

An axisymmetric magnetic field cannot be maintained via dynamo action.

The Ponomarenko dynamo

This is the simplest known kinematic dynamo. Consider plasma occupying all space; motion is confined to cylinder of radius a. Consider the MHD equation in the polar coordinates (r, θ, z).

The flow field is written:

$$\vec{V} = (0, r\Omega, U) \text{ for } r \leq a,$$

where Ω and U are constants. Consider incompressible flow $\nabla \cdot \vec{V} = 0$.

The dynamo equation is:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + \frac{\eta c^2}{4\pi} \nabla^2 \vec{B}.$$

We search for solution in the form:

$$\vec{B}(r, \theta, z, t) = \vec{B}(r) e^{\mathrm{i}m\phi - \mathrm{i}kz + \gamma t}$$

Using the vector formula:

$$\nabla\times(\vec{V}\times\vec{B}) = -\frac{1}{r}\frac{\partial}{\partial\phi}(r\Omega B_r) - \frac{\partial}{\partial z}(UB_r) = -\mathrm{i}(m\Omega - kU)B_r$$

$$(\nabla^2\vec{B})_r = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_r)\right] + \frac{1}{r^2}\frac{\partial^2 B_r}{\partial\phi^2} + \frac{\partial^2 B_r}{\partial z^2}$$

$$(\nabla^2\vec{B})_\phi = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\right] + \frac{1}{r^2}\frac{\partial^2 B_\phi}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial B_r}{\partial\phi} + \frac{\partial^2 B_\phi}{\partial r^2}$$

we derive the following equations for B_r and B_ϕ:

$$\gamma B_r = -\mathrm{i}(m\Omega - kU)B_r +$$

$$+\frac{c^2\eta}{4\pi}\left(\frac{d^2 B_r}{dr^2} + \frac{1}{r}\frac{dB_r}{dr} - \frac{m^2 + k^2r^2 + 1}{r^2}B_r - \frac{2\mathrm{i}m}{r^2}B_\phi\right)$$

$$\gamma B_\phi = r\frac{d\Omega}{dr}B_r - \mathrm{i}(m\Omega - kU)B_\phi +$$

$$+\frac{c^2\eta}{4\pi}\left[\frac{d^2 B_\phi}{dr^2} + \frac{1}{r}\frac{dB_\phi}{dr} - \frac{m^2 + k^2r^2 + 1}{r^2}B_\phi + \frac{2\mathrm{i}m}{r^2}B_r\right]$$

We keep $d\Omega/dr$ to evaluate the matching conditions at $r = a$.

Let

$$B_\pm = B_r \pm \mathrm{i}B_\phi;$$

$$y = r/a; \quad \tau_R = 4\pi a^2/c^2\eta;$$
$$q^2 = k^2a^2 + \gamma\tau_R + \mathrm{i}(m\Omega - kU)\tau_R;$$
$$s^2 = k^2a^2 + \gamma\tau_R.$$

Here τ_R is a typical diffusion time for magnetic field. Then,

$$y^2 B''_\pm + yB'_\pm - [(m \pm 1)^2 + q^2y^2]B_\pm = 0 \quad \text{for } y \le 1$$

$$y^2 B''_\pm + yB'_\pm - [(m \pm 1)^2 + s^2y^2]B_\pm = 0 \quad \text{for } y > 1$$

Solutions are Bessel's functions:

$$B_\pm = C_\pm I_{m\pm1}(qy)/I_{m\pm1}(q) \quad \text{for } y \le 1$$

$$B_\pm = D_\pm K_{m\pm1}(sy)/K_{m\pm1}(s) \quad \text{for } y > 1$$

where $C_\pm$ and $D_\pm$ are arbitrary constants. We find these and the eigenvalue γ (a dispersion relation) by matching the solutions at $y = 1$ (see [8], p.193).

In general, the dispersion relation is:

$$\gamma = \gamma(k, m, \tau_R, \Omega, U).$$

For $ka >> 1$ it becomes:

$$\gamma\tau_r \simeq e^{\mathrm{i}\pi/3}\left(\frac{m\Omega\tau_R}{2}\right)^{2/3} - k^2a^2 - \mathrm{i}(m\Omega - kU)\tau_r/2.$$

Dynamo takes place when $\mathrm{Re}(\gamma) > 0$:

$$\Omega\tau_R > \frac{2^{5/2}(ka)^3}{m}.$$

In agreement with the Cowling's theorem the axisymmetric dynamo $m = 0$ is impossible.

This is oscillatory dynamo because both $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are non-zero. The dynamo occurs whenever the flow is sufficiently rapid. The critical magnetic Reynolds number:

$$Re_m = \frac{\tau_R|V|}{a} = \frac{\tau_R\sqrt{\Omega^2a^2 + U^2}}{a}$$

is 17.7.

The dynamo action occurs when $Re_m > 17.7$.

Laboratory dynamo was achieved in the Riga experiment: stirring liquid sodium (best electroconducting fluid) at temperature 170°C by a propeller at $\Omega = 2000$ rpm.

Turbulent dynamo

Mean-Field Electrodynamics

Consider the MHD induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\vec{B} =$$

$$\equiv \nabla \times [\vec{v} \times \vec{B} - \frac{c^2}{4\pi\sigma}\nabla \times \vec{B}].$$

We seek a solution to the dynamo problem in terms of a mean magnetic field:

$$\vec{B} =< \vec{B} > +\vec{b},$$

where $\vec{b}$ is a fluctuating part of $\vec{B}$: $< \vec{b} >= 0$. Similarly, we consider a global and fluctuating motions:

$$\vec{v} =< \vec{v} > +\vec{u},$$

where $< \vec{u} >= 0$. Then, we separate the large-scale and fluctuating parts of the induction equation:

$$\frac{\partial (< \vec{B} > +\vec{b})}{\partial t} = \nabla \times [(< \vec{v} > +\vec{u}) \times (< \vec{B} > +\vec{b}) -$$

$$-\frac{c^2}{4\pi\sigma}\nabla \times (<\vec{B}> +\vec{b})].$$

The mean part is:

$$\frac{\partial <\vec{B}>}{\partial t} =$$

$$= \nabla \left[<\vec{v}> \times <\vec{B}> + <\vec{u}\times\vec{b}> - \frac{c^2}{4\pi\sigma}\nabla\times <\vec{B}> \right].$$

Subtracting the mean part from the total equation we get an equation for the fluctuating part:

$$\frac{\partial\vec{b}}{\partial t} = \nabla\times[<\vec{v}> \times\vec{b}+\vec{u}\times <\vec{B}> +\vec{u}\times\vec{b}- <\vec{u}\times\vec{b}> -$$

$$-\frac{c^2}{4\pi\sigma}\nabla\times\vec{b}].$$

The term

$$\mathcal{E} = <\vec{u}\times\vec{b}>$$

in the mean-field equation represents a mean electric field generated by fluctuating magnetic and velocity fields. If it is known we can solve the equation for $<\vec{B}>$. In principle, $\mathcal{E}$ must be calculated in terms of $<\vec{B}>$ using the equation for the fluctuating part, $\vec{b}$. However, in general, this is difficult.

However, we see that there is a linear relation between $\vec{b}$ and $<\vec{B}>$, and hence between $\mathcal{E}$ and $<\vec{B}>$. We write this relation as an expansion:

$$\mathcal{E} = \alpha < \vec{B} > -\beta\nabla\times < \vec{B} > +...$$

For almost isotropic turbulence:

$$\alpha \simeq \frac{1}{3} < \vec{u} \cdot \nabla \times \vec{u} > \tau,$$

$$\beta \simeq \frac{1}{3} < \vec{u} \cdot \vec{u} > \tau,$$

where τ is a characteristic correlation time.

The first term in the explanation for $\mathcal{E}$ is called α-effect, and $< \vec{u} \cdot \nabla \times \vec{u} >$ is ‘kinetic helicity’.

Then, the mean-field equation is:

$$\frac{\partial < \vec{B} >}{\partial t} = \nabla \times (< \vec{v} > \times < \vec{B} > +$$

$$+\alpha < \vec{B} > -(\eta + \beta)\nabla\times < \vec{B} >),$$

where $\eta = \frac{c^2}{4\pi\sigma}$ is magnetic diffusivity.

$$\eta + \beta \equiv \eta_t$$

is called turbulent diffusivity.

The α-term generates magnetic field providing the dynamo effect.

Reference: Equation for α and β are derived in A.R. Choudhuri, The Physics of Fluids and Plasmas, Cambridge Univ. Press, 1998.

Ω- and α-Effects

Consider an initially poloidal (in the r and θ directions, in spherical polar coordinates) fossil field subject to a differential azimuthal flow within the Sun. The field is stretched by the flow in the toroidal direction (the azimuthal direction) a process called the Ω effect.

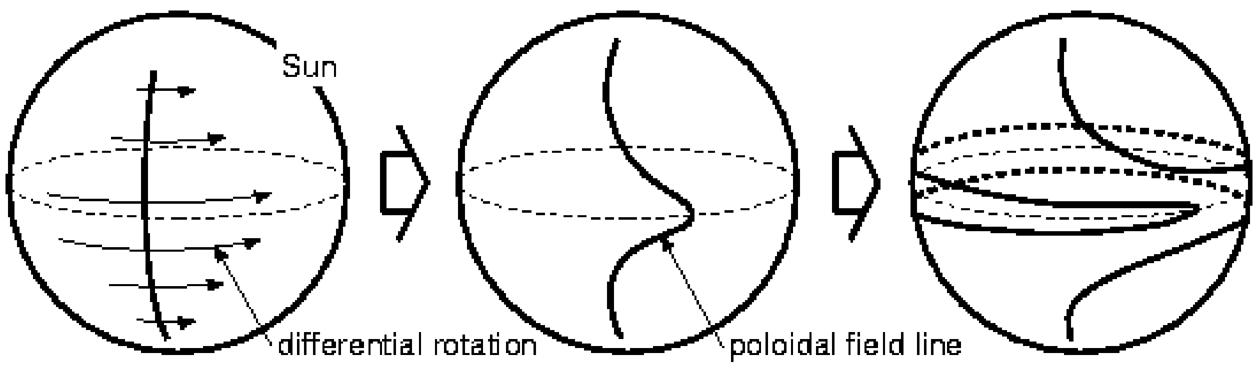

Figure 3: The Ω effect

To close the dynamo cycle, it is necessary to have a scheme whereby a poloidal field is reproduced from the stretched toroidal field bands produced by the Ω effect. This is argued to occur through cyclonic convection. A toroidal field line caught in a convection cell will be pulled into a loop.

The resulting rising Ω-loop (the name refers to the shape) will be twisted by the Coriolis force produced by the Sun's rotation. This twisting - which

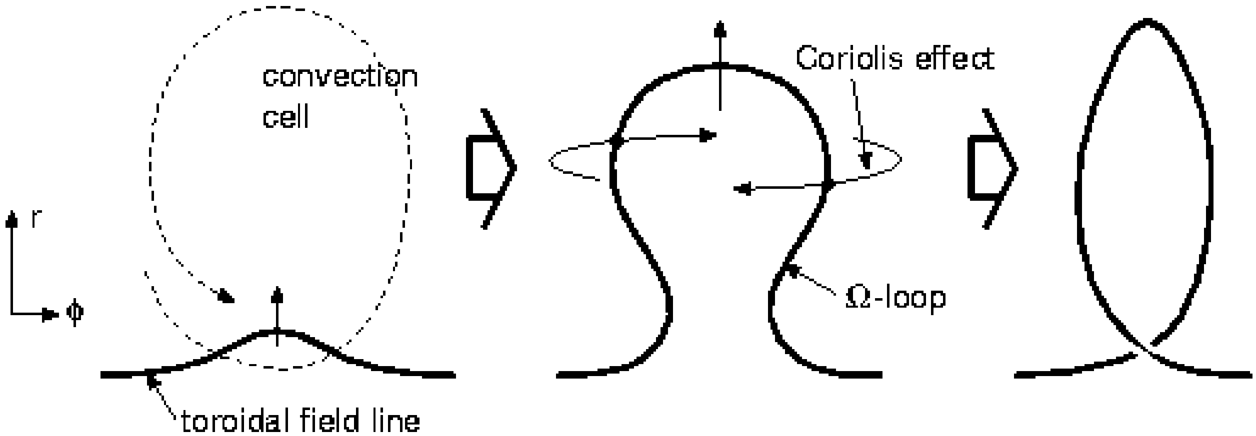

Figure 4: The α effect.

is known as the effect - leads to the top of the loop pointing in the poloidal direction, as shown in the right-hand panel of the above figure. If enough resulting poloidal field elements reconnect, a poloidal field will be reconstructed (with a reversed polarity from the original poloidal field).

Kinematic $\alpha\Omega$ Dynamo

In the astrophysical context, a dynamo is a fluid flow capable of sustaining a magnetic field indefinitely against Ohmic decay.

Consider a combined action of the α-effect and differential rotation. We shall assume that the kinetic helicity $\alpha(r, \theta)$ and angular velocity $\Omega(r, \theta)$ are

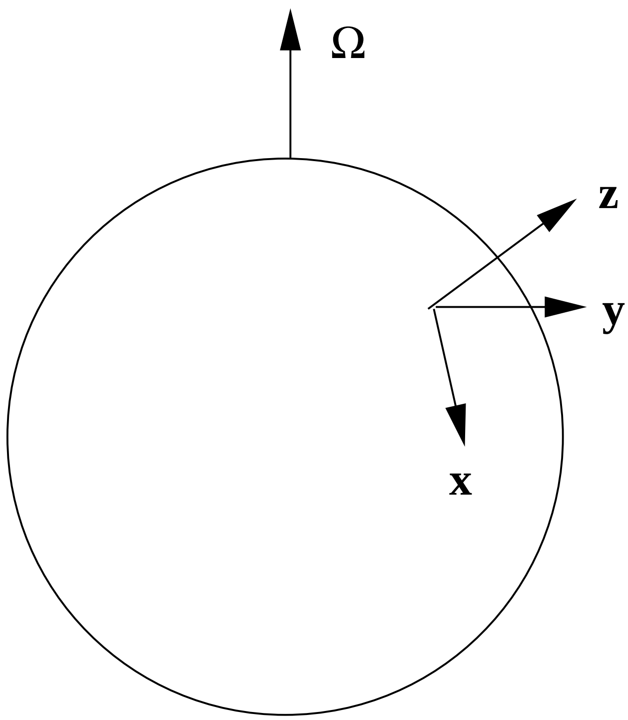

Figure 5: Coordinate system.

known functions. We consider the mean-field equation

$$\frac{\partial <\vec{B}>}{\partial t} = \nabla\times(<\vec{v}> \times <\vec{B}> +\alpha <\vec{B}> -\eta_t\nabla\times <\vec{B}>)$$

where $\vec{v}$ is the rotational velocity.

Consider this equation in cartesian coordinates in a small region of the Sun, so that axis x has direction along meridian, axis y is along the latitude, and axis z is perpendicular to the surface.

Magnetic field has two parts:

- toroidal $\vec{B}_t = B\vec{e}_y$

- poloidal $\vec{B}_p = (B_x, 0, B_z)$

The poloidal can be represented in terms of a vector-potential, A, which has only one component, A_y:

$$\vec{B} = \nabla \times \vec{A},$$

or

$$\vec{B}_p = \left(-\frac{\partial A}{\partial z}, 0, \frac{\partial A}{\partial x} \right).$$

The induction equation in terms of A is:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial \nabla \times \vec{A}}{\partial t} = \nabla \times [\vec{v} \times (\nabla \times \vec{A}) + \alpha \vec{B} -$$

$$-\eta_t \nabla \times \nabla \times \vec{A}].$$

From this equation we obtain equations for A and B:

$$\frac{\partial A}{\partial t} = \alpha B + \eta_t \nabla^2 A,$$

$$\frac{\partial B}{\partial t} = \frac{\partial v}{\partial z}\frac{\partial A}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial A}{\partial z} + \eta_t \nabla^2 B.$$

Since

$$v = r\Omega = (R + z)\Omega,$$

then

$$\frac{\partial v}{\partial z} = \Omega.$$

The derivative $\frac{\partial v}{\partial x}$ corresponds to the latitudinal differential rotation. However, in the first approximation we do not consider this term.

Thus, in the simple 1D case we have a system of two equations:

$$\frac{\partial A}{\partial t} = \alpha B + \eta \frac{\partial^2 A}{\partial x^2} \quad (1)$$

$$\frac{\partial B}{\partial t} = \Omega \frac{\partial A}{\partial x} + \eta_t \frac{\partial^2 B}{\partial x^2} \quad (2)$$

For constant coefficients α, Ω and η_t we can seek a solution in terms of periodic functions:

$$A = A_0 e^{-i\omega t + ikx},$$

$$B = B_0 e^{-i\omega t + ikx}.$$

Substituting these in the equations we obtain a linear system:

$$(-\omega + \eta_t k^2) A_0 - \alpha B_0 = 0 \quad (3)$$

$$(-\omega + \eta_t k^2) B_0 - \Omega i k A_0 = 0. \quad (4)$$

A non-zero solution exists if

$$(-i\omega + \eta_t k^2)^2 - ik\alpha\Omega = 0.$$

For $\alpha\Omega > 0$:

$$-i\omega + \eta_t k^2 = \pm\sqrt{i}\sqrt{k\alpha\Omega} = \pm\frac{1+i}{\sqrt{2}}\sqrt{k\alpha\Omega}$$

$$-i\omega = \left(-\eta_t k^2 + \sqrt{\frac{k\alpha\Omega}{2}}\right) + i\sqrt{\frac{k\alpha\Omega}{2}}.$$

This is a dispersion relation for dynamo waves.

Then the solution for toroidal magnetic field is:

$$B = B_0 \exp\left[\left(-\eta_t k^2 + \sqrt{\frac{k\alpha\Omega}{2}}\right)t + i\left(\sqrt{\frac{k\alpha\Omega}{2}}t + kx\right)\right].$$

It describes waves migrating poleward (towards negative x).

If we consider the case

$$\alpha\Omega < 0$$

then the propagation is the positive direction, towards the equator.

Magnetic field grows if

$$\frac{\alpha\Omega}{2\eta_t^2 k^3} > 1,$$

or in terms of Dynamo Number:

$$\mathcal{R}_D = \frac{\alpha\Omega R^3}{\eta_t^2},$$

$$\mathcal{R}_D (kR)^{-3} > 1.$$

By considering the dynamo equations in spherical coordinates one can show that the dynamo waves propagate along surfaces Ω=const, and the direction of propagation is given by the vector:

$$\alpha \nabla \Omega \times \vec{e}_\phi,$$

where $\vec{e}_\phi$ is an azimuthal unit vector.

The magnetic field growth is limited by back reaction on convection: magnetic field changes the properties of convection. This is modeled by "alpha-quenching":

$$\alpha = \frac{\alpha_0}{1 + (B/B_0)^2}.$$

This provides a stationary oscillatory solution.

It is believed that the solar dynamo operates in the tachocline at the low boundary of the convection zone because it is difficult to accumulate strong magnetic field in the convection zone.