MHD Approximation. Ohm's law.

([3], p. 169-183)

The kinetic equation for the distribution function $f(\vec{v}, \vec{r}, t)$ provides the most complete and universal description of plasma processes. In this approach we solve the kinetic equation for f and then calculate the macroscopic properties of plasma by integrating over the velocity distribution, e.g. density:

$$n(\vec{r}, t) = \int f(\vec{v}, \vec{r}, t) d^3\vec{v}$$

and velocity:

$$\vec{V}(\vec{r}, t) = \frac{1}{n} \int \vec{v} f(\vec{v}, \vec{r}, t) d^3\vec{v} \equiv < \vec{v} > .$$

If the macroscopic plasma properties vary much slower compared to the characteristic e-e and i-i collision times, then in small macroscopic elements of plasma the distribution functions are Maxwellian because of the frequent Coulomb collisions:

$$f_M(\vec{v}, \vec{r}, t) = n \left(\frac{m}{2\pi T} \right)^{3/2} \exp \left(-\frac{m(\vec{v} - \vec{V})^2}{2T} \right) .$$

In general, plasma properties vary in time and space: $n(\vec{r},t), \vec{V}(\vec{r},t), T(\vec{r},t)$. If at least one of these parameters of the distribution function is not a constant then such distribution is called *local thermodynamic equilibrium* LTE. This is reached in a characteristic time of e-e or i-i collisions. When all plasma parameters are constant in space and time and are the same for electrons and ions, then this is called *the complete thermodynamic equilibrium*. This requires much longer time because $\tau_{ei} \gg \tau_{ii} \gg \tau_{ee}$.

The complete thermodynamic equilibrium is usually not achieved in the presence of external forces.

If plasma is in LTE then the kinetic theory is not necessary. We describe the plasma processes in terms of n, V, and T. This gives a hydrodynamic theory. In the first approximation we assume that $f = f_M$. This means that we do not consider transfer processes for these parameters, that is diffusion, viscosity and heat conduction. The deviations from the Maxwellian distribution are important for the the transfer processes.

The processes of mass, momentum and energy transfer are relatively slow. Hence we can neglect these in the first approximation.

We derive equations for moments of the distribution function of particles of type a, $f(_a\vec{v}, \vec{r}, t)$

$$\langle ... \rangle = \frac{1}{n} \int (...) f d^3\vec{v}$$

starting from the kinetic equation in tensor notations:

$$\frac{\partial f_a}{\partial t} + \frac{\partial (v_\beta f_a)}{\partial x_\beta} + \frac{\partial}{\partial v_\beta}\left(\frac{F_{a,\beta}}{m_a} f_a\right) = St_{aa} + St_{ab}$$

where St_{aa} and St_{ab} are the collision terms ("integrals") for a-a and a-b collisions. Collision integrals describe how many particles disappear and appear in the six-dimensional phase space. Here a and b are either e or i. $F_{a,\beta}$ is the force acting on particles a:

$$\vec{F}_a = e_a \vec{E} + \frac{e_a}{c}[\vec{v} \times \vec{B}].$$

We multiply the kinetic equation by 1, $m_a v_\alpha$, and $m_a v_\alpha^2/2$ and integrate over the velocity space (getting zero, first and second moments).

The zero moment equation is:

$$\frac{\partial n_a}{\partial t} + \frac{\partial n_a V_{a\beta}}{\partial x_\beta} = 0,$$

where the integral over the third term in the kinetic equation is zero because it is reduced (due to the Gauss theorem) to a surface integral for an infinitely distant surface in the velocity space (where f is zero). The integral for the collision term is zero because of conservation of particles.

We can write this equation in the vector form:

$$\frac{\partial n_a}{\partial t} + \nabla(n_a \vec{V}_a) = 0.$$

Calculating the first moment, we get:

$$\frac{\partial}{\partial t}(m_a n_a V_{a\alpha}) + \frac{\partial}{\partial x_\beta}(m_a n_a < v_{a,\alpha} v_{a,\beta} >) -$$

$$-e_a n_a (E_\alpha + \frac{1}{c}[\vec{V}_a \times \vec{B}]_\alpha) = R_{ab,\alpha}$$

where

$$R_{ab,\alpha} = \int m v_\alpha St_{ab} d^3\vec{v}$$

St_{aa} does not contribute to the momentum source because of the conservation of momentum for particle collisions of the same type. Also, because of the conservation of the total momentum in collisions between a and b particles:

$$\vec{R}_{ba} = -\vec{R}_{ab}.$$

In the third term we used integration by parts:

$$\int v_\alpha \frac{\partial}{\partial v_\beta}\left[(eE_\beta + \frac{e}{c}[\vec{v}\times\vec{B}]_\beta)f\right] d^3\vec{v} =$$

$$\int \frac{\partial}{\partial v_\beta}\left[v_\alpha(eE_\beta + \frac{e}{c}[\vec{v}\times\vec{B}]_\beta)f\right] d^3\vec{v}$$

$$-\int (eE_\beta + \frac{e}{c}[\vec{v}\times\vec{B}]_\beta)\underbrace{\frac{\partial v_\alpha}{\partial v_\beta}}_{\delta_{\alpha\beta}} f d^3\vec{v}.$$

Finally for the second moment ($\int(...)m_a v_a^2/2d^3\vec{v}$) using again integration by parts we get:

$$\frac{\partial}{\partial t}\left(\frac{m_a n_a}{2} < v_a^2 >\right) + \frac{\partial}{\partial x_\beta}\left(\frac{m_a n_a}{2} < v^2 v_\beta >\right) -$$

$$-e_a n_a \vec{E}\cdot\vec{V} = \int \frac{m_a v_a^2}{2} St_{ab} d^3\vec{v}$$

Separate the particle velocity $\vec{v}$ into macroscopic $\vec{V}$ and chaotic $\vec{v}'$ components (for simplicity we omit the particle index a):

$$\vec{v}' = \vec{v} - \vec{V}$$

where

$$< \vec{v}' >= 0$$

$$\left\langle \frac{mv'^2}{2} \right\rangle = \frac{3}{2}T.$$

Then, in the first-moment equation:

$$\langle v_\alpha v_\beta \rangle = V_\alpha V_\beta + \left\langle v'_\alpha v'_\beta \right\rangle$$

$$\frac{\partial}{\partial t}(mnV_\alpha) + \frac{\partial}{\partial x_\beta}(mnV_\alpha V_\beta) + \frac{\partial}{\partial x_\beta} mn \left\langle v'_\alpha v'_\beta \right\rangle - F_{M\alpha} =$$

$$= R_\alpha$$

where

$$F_{M\alpha} = en\left(\vec{E} + \frac{1}{c}\left[\vec{V} \times \vec{B}\right]\right)_\alpha .$$

Consider the first two terms:

$$mn\frac{\partial V_\alpha}{\partial t} + \underbrace{mV_\alpha \frac{\partial n}{\partial t} + mV_\alpha \frac{\partial nV_\beta}{\partial x_\beta}}_{=0,\text{because of continuty eq.}} + \underbrace{mnV_\beta \frac{\partial V_\alpha}{\partial x_\beta}}_{mn(\vec{V}\cdot\nabla)\vec{V}} .$$

The third term is the pressure tensor:

$$P_{\alpha\beta} = \int m v'_\alpha v'_\beta f d^3\vec{v} = mn \left\langle v'_\alpha v'_\beta \right\rangle = p\delta_{\alpha\beta} + \pi_{\alpha\beta},$$

where p is the scalar pressure:

$$p = mn < {v'_x}^2 >= mn < {v'_y}^2 >= mn < {v'_z}^2 >=$$

$$= \frac{1}{3} mn < v'^2 > .$$

For real systems the distribution function can be quite anisotropic.

In the isotropic case, the momentum equation is:

$$mn \frac{dV_\alpha}{dt} = -\frac{\partial p}{\partial x_\alpha} + en[\vec{E} + \frac{1}{c}\vec{V} \times \vec{B}]_\alpha + R_\alpha,$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_\beta \frac{\partial}{\partial x_\beta}.$$

This is so-called full (or material) derivative.

Now, we consider the second term of the second-moment (energy) equation:

$$\left\langle \frac{v^2}{2} v_\beta \right\rangle = \frac{V^2}{2} V_\beta + V_\alpha \left\langle v'_\alpha v'_\beta \right\rangle + \frac{1}{2}\left\langle v'^2 \right\rangle V_\beta + \left\langle \frac{v'^2}{2} v'_\beta \right\rangle =$$

$$= \left(\frac{V^2}{2} + \frac{5}{2} \frac{p}{mn} \right) V_\beta + \frac{V_\alpha \pi_{\alpha\beta}}{mn} + \left\langle \frac{1}{2} v'^2 v'_\beta \right\rangle .$$

Hence, for the energy equation we get:

$$\frac{\partial}{\partial t} \left(\frac{nmV^2}{2} + \frac{3nT}{2} \right) +$$

$$+ \frac{\partial}{\partial x_\beta} \left[\left(\frac{nmV^2}{2} + \frac{5nT}{2} \right) V_\beta + \pi_{\alpha\beta} V_\alpha + q_\beta \right] =$$

$$= en\vec{E} \cdot \vec{V} + \vec{R} \cdot \vec{V} + Q,$$

where

$$q_\beta = \int \frac{mv'^2}{2} v'_\beta f d^3\vec{v} = mn \left\langle \frac{v'^2}{2} v'_\beta \right\rangle$$

is called the heat flux.

$$Q = \int \frac{mv'^2}{2} St_{ab} d^3\vec{v}$$

is the heat transfer rate from particles b to particles a.

The term $en\vec{E} \cdot \vec{V}$ is the work of the electric field on charged particles (in unit volume and in unit time); $\vec{R} \cdot \vec{V}$ is the work of the friction force $\vec{R}$.

Using mathematical identities and the momentum equation the energy equation can be reduced to:

$$\frac{3}{2}n\frac{dT}{dt}+p\frac{\partial V_\beta}{\partial x_\beta}=-\frac{\partial q_\beta}{\partial x_\beta}+Q+\vec{R}\vec{V}.$$

MHD equations

Consider two-fluid continuity and momentum equations for electrons and ions:

$$\frac{\partial n_e}{\partial t} + \nabla(n_e \vec{V}_e) = 0$$

$$\frac{\partial n_i}{\partial t} + \nabla(n_i \vec{V}_i) = 0$$

$$m_e n_e \frac{d\vec{V}_e}{dt} = -\nabla p_e - e n_e[\vec{E} + \frac{1}{c}\vec{V}_e \times \vec{B}] + \vec{R}_{ei}$$

$$m_i n_i \frac{d\vec{V}_i}{dt} = -\nabla p_i + Z e n_i[\vec{E} + \frac{1}{c}\vec{V}_e \times \vec{B}] + \vec{R}_{ie}$$

where the drag force $\vec{R}_{ei} = -\vec{R}_{ie}$. It can be approximated as $R_{ei} \approx m_e n_e(\vec{V}_i - \vec{V}_e)\overline{\nu}_{ei}$. The plasma is described in terms of two mutually penetrating fluids.

Quite often plasma can be considered as a single fluid. Consider a plasma with singly ionized ions:

$$Z = 1, \quad n_e = n_i = n.$$

Introduce mass density ρ and hydrodynamic velocity $\vec{V}$:

$$\rho = n_i m_i + m_e n_e = n(m_i + m_e)$$

$$\vec{V} = \frac{1}{\rho}(n_i m_i \vec{V}_i + n_e m_e \vec{V}_e) =$$

$$= \frac{m_i \vec{V}_i + m_e \vec{V}_e}{m_i + m_e}$$

Then multiplying the continuity equations by m_e and m_i and taking their sum we get:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0$$

- the conservation of mass.

Then, we take sum of the momentum equations:

$$\rho \frac{d\vec{V}}{dt} = -\nabla p + \frac{1}{c}\vec{j} \times \vec{B},$$

where

$$\vec{j} = e(n_i \vec{V}_i - n_e \vec{V}_e) = e n_e (\vec{V}_i - \vec{V}_e)$$

is the electric current density,

$$p = p_i + p_e$$

is the total pressure, $\vec{j} \times \vec{B}/c$ is so-called ponderomotive force.

Now, we consider the difference of the momentum equations multiplied by m_i and m_e:

$$m_i m_e n \frac{d}{dt}(\vec{V}_i - \vec{V}_e) = -m_e \nabla p_i + m_i \nabla p_e + en(m_i + m_e)\vec{E} +$$

$$+\frac{en}{c}(m_e \vec{v}_i + m_i \vec{V}_e) \times \vec{B} - (m_i + m_e)\vec{R}_{ei}.$$

Express in terms of ρ and $\vec{j}$:

$$m_e \vec{V}_i + m_i \vec{V}_e =$$

$$= m_e \vec{V}_i + m_i \vec{V}_e + m_i \vec{V}_i - m_i \vec{V}_i + m_e \vec{V}_e - m_e \vec{V}_e =$$

$$= \frac{\rho}{n}\vec{V} - (m_i - m_e)(\vec{V}_i - \vec{V}_e) =$$

$$= \frac{\rho}{n} - (m_i - m_e)\frac{\vec{j}}{ne}$$

$$\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right) = -m_e \nabla p_i + m_i \nabla p_e + e\rho \vec{E} +$$

$$+\frac{e\rho}{c}\vec{V} \times \vec{B} - (m_i - m_e)\frac{\vec{j} \times \vec{B}}{c} -$$

$$-\frac{(m_i + m_e)m_e}{e}\vec{j}\overline{\nu}_{ei} \approx 0$$

for slow motions. Assume $m_e \ll m_i$, then

$$m_i \nabla p_e + e\rho(\vec{E} + \frac{1}{c}\vec{V} \times \vec{B}) -$$

$$-\frac{m_i}{c}\vec{j} \times \vec{B} - \frac{m_i m_e \overline{\nu}_{ei}}{e}\vec{j} = 0$$

Then,

$$\vec{E} + \frac{1}{c}\vec{V} \times \vec{B} = \frac{1}{enc}[\vec{j} \times \vec{B} - \nabla p_e] + \underbrace{\frac{m_e \overline{\nu}_{ei}}{e^2 n}}_{1/\sigma}\vec{j},$$

where σ is electrical conductivity.

Hence, we obtain a **generalized Ohm's law**:

$$\vec{j} = \sigma[\vec{E} + \frac{1}{c}\vec{V} \times \vec{B}] - \frac{\sigma}{enc}[\underbrace{\vec{j} \times \vec{B}}_{\text{Hall effect}} - \nabla p_e].$$

In magnetic field, in addition to electric current along the electric field there is also current that is perpendicular to $\vec{E}$ (Hall's current).

Application of the Ohm's law depends on boundary conditions because electric charges may appear.

Consider the case $V = 0$, and $\nabla p_e = 0$.

$$\vec{j} = \sigma\vec{E} - \frac{\sigma}{enc}[\vec{j} \times \vec{B}]$$

in a simple geometry:

$$\vec{B} = (0, 0, B) \quad \text{and} \quad \vec{E} = (E, 0, 0,)$$

Then,

$$j_x = \sigma E_x - \frac{\sigma B}{enc} j_y$$

$$j_y = \frac{\sigma B}{enc} j_x$$

$$j_z = 0$$

We get

$$j_x = \sigma E_x - \frac{\sigma^2 B^2}{(enc)^2} j_x$$

$$j_x = \frac{\sigma E_x}{1 + \left(\frac{\sigma B}{enc}\right)^2} = \sigma_\perp E_x$$

$$j_y = \frac{\alpha \sigma E_x}{1 + \alpha^2} = \sigma_H E_x$$

where

$$\alpha = \frac{\sigma B}{enc} = \frac{eB}{mc\overline{\nu}_{ei}} = \frac{\omega_c}{\overline{\nu}_{ei}}$$

$\omega_c = eB/mc$ is the cyclotron frequency. Hence, the electrical conductivity in plasma with magnetic field is anisotropic.

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

When $\alpha \gg 1$ the electrical conductivity perpendicular to the magnetic field is significantly reduced. However, the Joule heating $\vec{j}\vec{E}$ is not affected.

Consider a cylindrical plasma with magnetic field $\vec{B}$ along the axis z and electric field along the radius.

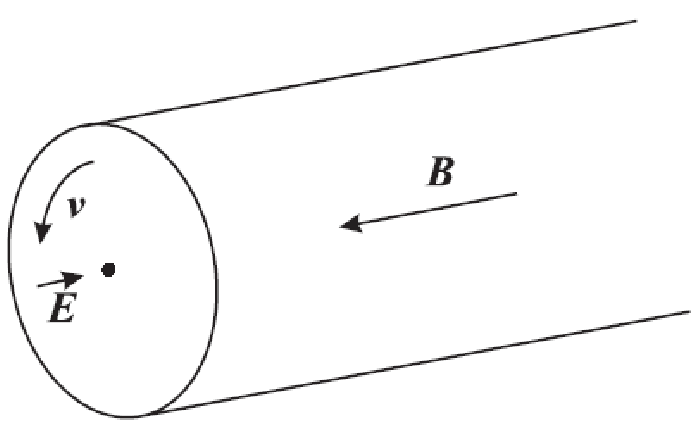

Figure 1: Hall effect in cylindrical plasma

In this case, the solution to the MHD equations is:

$$\vec{V}_e = \vec{V}_i = \frac{c}{B^2}[\vec{E} \times \vec{B}]$$

$$\vec{j} = 0$$

The plasma simply rotates around the cylinder axis. The electric current is zero for non-zero electric field. This means that plasma behaves as a dielectric.

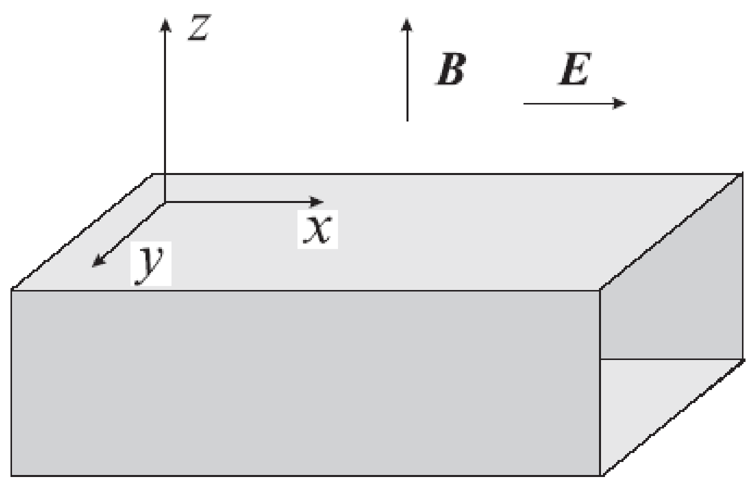

Figure 2: Hall effect in a plasma channel.

However, if we consider a plasma channel along x-axis with $\vec{B}$ along z-axis and $\vec{E}$ along x axis then the Hall motion in the y-direction is not possible because of the channel boundaries. This leads to a pressure gradient that cancels the Hall effect. Then, the momentum equation for electrons is

$$-en_e\vec{E} + \vec{R}_{ei} = 0$$

$$-en_e\vec{E} + m_e n_e(\vec{V}_i - \vec{V}_e)\overline{\nu}_{ei}$$

which leads to

$$\vec{j} = \sigma\vec{E}$$

Magnetic field does not effect electric current in this case.