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COURSE 7

LINEAR ADIABATIC STELLAR PULSATION

Douglas O. Gough

Institute of Astronomy and
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, CB3 0HA, UK
and
Joint Institute for Laboratory Astrophysics,
University of Colorado, Boulder CO 80309, USA

J.-P. Zahn and J. Zinn-Justin, eds.
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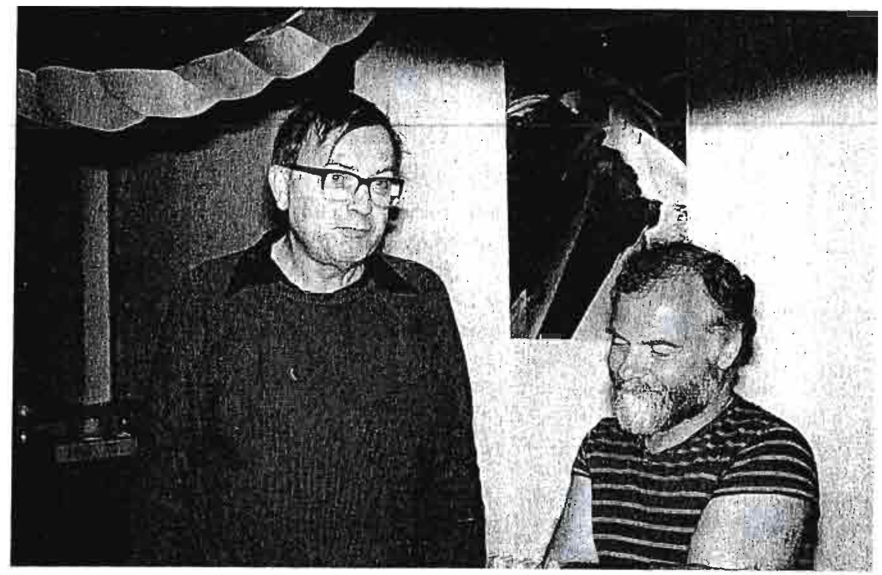
Dynamique des fluides astrophysiques
Astrophysical fluid dynamics



Contents

1. Introduction

These lectures address but a part of the theory of stellar pulsation: the linearized adiabatic theory of acoustic gravity modes. My discussion is concerned with the dominant aspects of the dynamics of small oscillations, and therefore provides an introduction to the wider theory. Thus, I will consider the determination of the frequencies of the modes of oscillation, but I will not address the important issue of how the modes are driven. Consequently, I will not provide any assistance for judging which stars are likely to pulsate.

Acoustic and gravity modes are probably the most common modes of oscillation of stars; they are certainly the most commonly studied. They are the simplest to describe, and can exist in an otherwise static, spherically symmetrical star.

For such modes, both the terms pulsation and oscillation have been used in the literature, and I will use them interchangeably. I will consider spherically symmetrical stars first, and do so in greatest detail. I begin by discussing the simplest of their oscillations, namely the spherically symmetric (radial) pulsations; this will give me the opportunity to introduce some of the mathematical techniques I require, without confusing matters with unnecessary complexity. Then I will consider so-called nonradial oscillations, which are oscillations that are not purely radial. I will offer but a taste of the complications that are introduced once the star is considered to be intrinsically aspherical; only a small asphericity will be permitted, which forms in some sense a small though significant perturbation to the modes of oscillation.

Deviations from spherical symmetry of the equilibrium structure of a star are brought about by agents ignored in the spherically symmetrical theory, such as rotation, a magnetic field or convection. Associated with each such agent, provided it is globally stable, is a new class of oscillations: inertial oscillations, resonant Alfvén waves and convective waves, e.g. I do not discuss any of these.

Several techniques are available for determining the oscillation eigenfrequencies; for accurate frequencies of realistic stellar models they all require resorting in the end to numerical computation. I will not discuss numerical methods at all in these lectures. Instead I will concentrate on asymptotic expansions which yield analytical formulae, and consequently more readily provide physical insight. That is not to imply that numerical solutions are of lesser scientific value. Indeed, in practice it is nearly always a numerical solution that must be compared with mea-

surement in order to make precise inferences about the structure or dynamics of a star. However, it is usually the case that the nature of the comparison is motivated by the form of an approximate asymptotic formula, and the significance of the result is appreciated only in the light of the physical insight the formula provides. For this reason, the development of analytical results, despite their lack of precision, is essential to the progress on the subject.

One cannot appreciate the small oscillations of a system without some understanding of the nature of the equilibrium state. Therefore, at the school these lectures were preceded by a short introductory course on the theory of stellar structure and evolution. Notes on that course are not reproduced here, because the material is more than adequately covered in the textbooks on the subject.

These lectures do not constitute a balanced review of the subject, but instead concentrate on a few specific aspects of current interest. There are some useful modern textbooks on stellar pulsation to balance my bias, most notably those by Unno et al. (1979) and Cox (1980). The first is closer in both style and content to these lectures, and much of what I have to say can be found, sometimes in a somewhat different form, in it. It deals more widely with the subject than I do here, and therefore is an invaluable explanatory accompaniment to these notes. The second book is more descriptive, dwelling more on the nature of the physical ideas than on their precise consequences; it therefore provides complementary background reading. I must not fail to recommend the now classical review by Ledoux and Walraven (1958); although it is some thirty years since it was written, the article still gives a wealth of useful information that is still relevant today.

Although stellar pulsation is a fluid-dynamical phenomenon, little prior knowledge of fluid dynamics is required to understand these lectures. The linearized stability equations that form the basis of the subject are easily derived from the equations of momentum and mass conservation; for understanding the manipulations that follow, it is more useful to have some knowledge of the theory of linear differential equations.

Partly because this subject is an amalgam of other disciplines, mainly the theory of stellar structure and the theory of waves, and partly because I use some quite distinct approaches to discuss the subject, such as the separation of variables in section 5 leading directly to a set of ordinary differential equations with respect to each of the spherical polar coordinates and the direct asymptotic attack on the partial differential equations in section 8 leading to ordinary differential equations initially along ray paths, I will quickly run out of letters in the alphabet to represent quantities unambiguously. Of course I could make unambiguous distinctions by adorning the symbols with accouterments, and indeed at times I have been moved to do so, but in order not to complicate the notation unnecessarily and thereby conceal the import of the equations, I have often preferred to give of some of the symbols a duplicate meaning, at what I hope is only a slight risk of misunderstanding. Except in section 7, summation over repeated indices that denote components of a vector or tensor, is assumed throughout.

I start these lectures by discussing the nature of the fluid that constitutes the star; I introduce the equations of fluid dynamics, ignoring dissipation without discussion. I then record in section 2 the relations defining the hydrostatic support of the background equilibrium state that I will need immediately, and proceed in section 3 to the formal linearized perturbation equations. As I have remarked already, the perturbation equations are discussed first for the geometrically simple radial pulsations; I do this in section 4. Boundary conditions are obtained, orthogonality of the eigenfunctions is established and a variational principle is derived from which I determine a bound on the pulsation frequency. My main interest is in the dynamics of the stellar interior, where the energy in the oscillations mainly resides. The outer boundary condition therefore requires a study of the forced oscillations of the stellar atmosphere, which is considered to lie outside the surface enclosing the dynamically most interesting part of the star. The very centre of the star is merely a coordinate boundary, and conditions there are determined by regularity. Section 4 is concluded by a discussion of asymptotically high-frequency oscillations, whose eigenfunctions have the character of waves. To a large extent the immediately following discussion in section 5 of nonradial oscillations parallels its radial counterpart. Some care is taken to cast the equations in a separable form that resembles the more familiar oscillation equations established originally in plane geometry, since then one is in a better position to investigate how the spherical geometry influences the modes. The asymptotic methods in the previous section are applied to these equations, to establish what are perhaps surprisingly simple integral expressions for the eigenfrequencies. The asymptotic analysis is wave-like, and to complement this approach, a procedure based on representing the complete eigenfunction as a superposition of resonant, locally plane waves, which was originally developed for quantum theory and is now often referred to as semi-classical quantization, is discussed in section 8. In principle this approach is in some respects more powerful, because it does not rely on the necessity to separate coordinates at the outset. However, since in practice I never consider other than small deviations of the basic state from spherical symmetry, the eigenfunctions are always almost separable and the full generality of the approach is not utilized. The results I establish from it are completely equivalent to those that can be obtained by perturbing the separable solutions, which is described in section 7. One of the motivations for paying so much attention to asymptotic solutions is that the Sun is known to be oscillating in many acoustic modes that can be quite accurately described by asymptotic theory. Moreover, some Ap stars have been observed to oscillate in high-frequency acoustic modes too. One of the very convenient properties of the asymptotic eigenvalue equations established in section 5, is that in their simplest forms they can be inverted analytically, providing explicit constraints on the stratification of the background state. This inversion is presented in section 6. In the case of acoustic modes, one formally obtains the speed of sound as a function of radius; such an inversion of frequency data has already been carried out on observed frequencies of the Sun. In the case of gravity modes one obtains a functional constraint on the buoyancy frequency; in the future this relation may be useful for setting limits on the internal stratification of white dwarfs. Stellar inverse theory is in its infancy. However, the appreciation of the dependence of oscillation frequencies on the stratification of a star that can be acquired by the so-called forward methods described in these lectures, will no doubt provide a substantial contribution to the future development of that theory.

1.1. The fluid

The fluid will be regarded as a continuum mixture of gas and radiation. Thus there is an equation of state

$$p = p(\rho, T; \boldsymbol{X}), \tag{1.1.1}$$

relating the pressure, p, to the density, ρ, and the temperature, T. The relation also depends on the composition of the gas, which I denote as $\boldsymbol{X}$, each component X_i of which is the relative abundance by mass of the chemical element i; of greatest interest is the dependence on the most abundant elements, hydrogen and helium, and throughout this course I will, for simplicity, replace $\boldsymbol{X}$ by the component $X_1 =: X$, which is the hydrogen abundance, [H]. The helium abundance $X_2 =: Y := [^4\text{He}]$ is then given by $Y = 1 - X - Z$, where Z is the sum of the abundances, [Hea], of all the heavier elements.

For the purposes of these lectures it will not be necessary to specify the equation of state explicitly. It is perhaps worth making the obvious remark, however, that the accuracy to which one can calculate the dynamical oscillation frequencies of any given theoretical model of a star is limited by the accuracy to which the equation of state is known. Observations of solar oscillations have now reached the point where frequencies are quoted to better than 1 part in 10^4 (e.g. Jiminez et al. (1987)), which surpasses by a large margin our ability to compute an accurate equation of state. By the standards of these observations, the physics of dense plasmas is very poorly understood: we are not yet in a position to take into account, with sufficient accuracy, the influence of neighbouring particles on the bound states of neutral atoms or compound ions, e.g., and thereby calculate the appropriate ensemble average of the energy of their interaction, which it is necessary to know in order to calculate the compressibility of the fluid. But that does not mean that we cannot make progress with our study of stellar oscillations. It behoves us merely to be aware that the results of our calculations really depend on certain thermodynamic derivatives that appear in constitutive relations, and not directly on what one sometimes considers to be the more basic thermodynamic properties of the gas, such as temperature and composition. The most important of these is the first adiabatic exponent

$$\gamma := \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s , \tag{1.1.2}$$

the derivative being taken at constant specific entropy, s. (I do not use a suffix on γ to distinguish it from the other adiabatic exponents, since this quantity will be the only adiabatic exponent used in the course.)

To guide ideas, it is often helpful to think in terms of a perfect gas. It is a good first approximation for main-sequence stars. The equation of state is

$$p = \frac{\mathcal{R}\rho T}{\mu}, \tag{1.1.4}$$

where $\mathcal{R}$ is the gas constant and μ is the reciprocal of the number of particles per hydrogen-atom mass of the gas, the so-called mean molecular mass (or weight). Since $Z \ll 1$, we have for the completely unionized state:

$$\mu^{-1} = X + \tfrac{1}{4}Y + \mu_0^{-1}Z \simeq 1 - \tfrac{3}{4}Y, \tag{1.1.5}$$

where μ_0 is the mean atomic mass of the heavy elements, and when the hydrogen and helium are completely ionized:

$$\mu^{-1} \simeq 2X + \tfrac{3}{4}Y \simeq 2 - \tfrac{5}{4}Y. \tag{1.1.6}$$

Taking $Y = 0.25$, which is roughly the value outside the energy-generating cores of main-sequence stars, yields $\mu \simeq 1.2$ and 0.6 for the unionized and ionized state, respectively. Thus, the ratio $p/\rho T$ changes by a factor two along the ionization zones in the outer envelopes of stars. Outside the ionization zones $\gamma \simeq \frac{5}{3}$. When abundant elements are in a state of partial ionization, work is used to ionize the gas when it is compressed adiabatically, so the temperature rise is less than it would have been otherwise. The pressure rise is also smaller, despite the decrease in μ. Thus, γ is less than $\frac{5}{3}$ (though it never drops as low as unity, the value it would have taken were T to have remained precisely constant on compression). Figure 1 illustrates how $\rho T/p$ and γ vary along the ionization zones of a model of the envelope of the Sun.

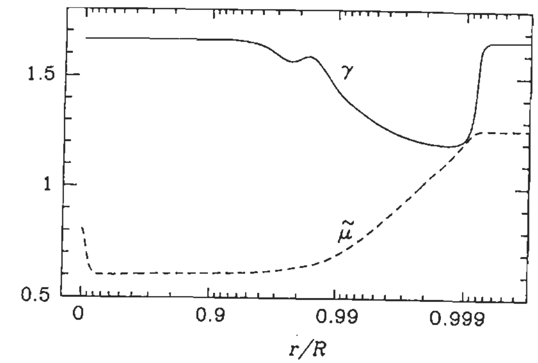


Fig. 1. The adiabatic exponent γ and the quantity $\tilde{\mu} = \mathcal{R}\rho T/p$ plotted against r/R for a solar model, where r is the radial coordinate and R is the radius of the photosphere. The equation of state used to compute the model is more complicated than eq. (1.1.4), since it takes some of the electrostatic interactions between neighbouring particles into account, and so $\tilde{\mu}$ is not precisely μ. (Indeed, some approaches to studying the equation of state are such that when interactions between particles are significant even the concept of μ is not well defined.)

1.2. Equations of motion

The Eulerian momentum equation, ignoring viscosity, for a fluid moving with velocity $\boldsymbol{u}$ is

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{g}\rho + \boldsymbol{\mathcal{F}}, \tag{1.2.1}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{1.2.2}$$

is the Lagrangian time derivative, t the time, $\boldsymbol{g} = \nabla\Phi$ is the acceleration due to gravity and $\boldsymbol{\mathcal{F}}$ is the body force due to all agents except gravity (such as a possible magnetic field). The gravitational potential, Φ, satisfies Poisson's equation:

$$\nabla^2\Phi = -4\pi G\rho. \tag{1.2.3}$$

The equation of conservation of mass is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \,\mathrm{div}\, \boldsymbol{u} = \frac{\partial \rho}{\partial t} + \mathrm{div}\, \rho\boldsymbol{u} = 0. \tag{1.2.4}$$

Throughout these lectures the adiabatic approximation will be used. This determines uniquely the relation between variations in pressure and density:

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = c^2\frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{1.2.5}$$

These equations must be supplemented by the equation of state. For adiabatic motion, that is all that is required to close the equations, once the basic state, about which the star pulsates, is defined. Indeed, the sole quantity derived from the equation of state that is needed to study adiabatic perturbations, and on which the stratification of much of the basic state might not directly depend is the thermodynamic derivative

$$\gamma = \gamma(p, \rho, X). \tag{1.2.6}$$

We notice immediately from the discussion at the end of section 1.1 that, except in the ionization zones of abundant elements, γ is essentially independent of chemical composition. Therefore, since the ionization zones occupy a relatively small fraction of the volume of the star, the pulsation frequencies, ω, should not be very sensitive to X. That simplifies the task of predicting ω. But conversely, it renders it difficult to determine X from a knowledge of the oscillation frequencies of a star.

To determine the solution to the differential equations describing the motion it is also necessary to specify boundary conditions. In all cases it will be assumed that one can define a "surface" of the star outside of which there is negligible material. Boundary conditions are then essentially that there is no stress exerted on the boundary, or, more realistically, that there is no inward flow of energy across the boundary. The latter condition selects the so-called "causal" solution. In practice, it will often be found convenient to apply boundary conditions deeper in the stellar atmosphere, obtained by matching the oscillation in the interior of an appropriate surface with an analytical approximation to the causal solution outside it. This will be discussed in more detail later.

2. The equilibrium state

I will be limiting my discussion to the linearized theory of small perturbations of a basic state. That basic state is provided by the theory of stellar evolution. The time scales characterizing the evolution of a star are normally much longer than the pulsation periods, and for most stars they are also much longer than the rates of growth or decay of the pulsations. Therefore, the basic state will be considered

to be independent of time: an equilibrium state. It satisfies the time-independent version of the equations in the previous section.

For most of these lectures I will assume the star to be nonmagnetic and nonrotating. I also ignore the possibility of any large-scale circulatory motion and assume convective motion to be on a spatial scale that is so small that I can average over the inhomogeneities to obtain a smoothed-out description of the stratification. Thus, I set $\boldsymbol{u} = 0$ and $\mathcal{F} = 0$ in eq. (1.2.1). The equilibrium state is then spherically symmetrical, satisfying the hydrostatic equation:

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} + g_0\rho_0 = 0, \tag{2.1}$$

where r is a radial coordinate and the subscript zero denotes equilibrium value. The quantity g_0 is the magnitude of the gravitational acceleration, which is obtained by integrating eq. (1.2.3); since the state is spherically symmetrical, this yields

$$g_0 = \frac{Gm_0}{r^2}, \tag{2.2}$$

where m_0 is the mass enclosed in a sphere with constant r:

$$m_0(r) = 4\pi\int_0^r \rho_0(r')r'^2\,\mathrm{d}r'. \tag{2.3}$$

Equations (2.1)–(2.3) and the equation of state (1.1.1), together with appropriate boundary conditions, are not sufficient to determine the basic state. That state is determined as a solution of the full equations of stellar evolution, which I do not discuss here. Nevertheless, the hydrostatic balance expressed by eqs. (2.1)–(2.3) is all that is necessary for the validity of the linearized perturbation analysis I will be discussing. Since I have in mind studying the oscillations about a time-independent state, the temporal behaviour of the perturbations is sinusoidal with frequency ω.

I should point out that it is possible to study the temporal development of perturbations about an evolving state which does not satisfy all the steady-state equations. In particular, attention has been devoted in the past to the stability to pulsation of stars that are out of thermal balance. A proper analysis involves the use of a more sophisticated theory than that which I propose to discuss in this course.

One property of the equilibrium state to which I wish to draw attention is the behaviour of the variables near the centre. Equation (2.3) implies

$$m_0 \sim \tfrac{4}{3}\pi\rho_{00}r^3 \quad \text{as } r \to 0, \tag{2.4}$$

where ρ_{00} is the density at the centre. Hence, from eq. (2.2), $g_0 \propto r$ and the pressure gradient vanishes at $r = 0$ as a consequence of ρ_0 being finite. Equation (2.1) can be integrated to yield

$$p_0 \sim p_{00} - \tfrac{2}{3}\pi G \rho_{00}^2 r^2. \tag{2.5}$$

One can analyze the development of ρ_0 and the temperature T_0 away from the origin in a similar way, using the energy equation, which I have not written down, and the equation of state, coupled with the expansion (2.5). Let us consider first the case where the central regions of the star are stable to convection. As a consequence of there being no point source of heat at the centre of the star and the heat flux being proportional to the temperature gradient, one finds that the gradients of temperature and density also vanish at $r = 0$, provided that the composition gradients vanish too. At zero age, the composition is usually presumed to be uniform. It is subsequently modified by nuclear reactions, whose rates, being a function of ρ_0, T_0 and composition, must initially have zero gradient at $r = 0$. Consequently the rates of change of X_0 and Z_0 have zero gradient at $r = 0$ too, and therefore p_0, ρ_0, T_0 and all other thermodynamic variables have zero gradient at $r = 0$ for all times. When the core is convective, the stellar material is homogenized by the motion, and the stratification is essentially adiabatic; therefore, it follows again that density and temperature have zero gradient at the centre.

3. Linearized equations

It is common practice to describe the oscillations about the equilibrium state discussed in the previous section, in terms of the displacement, $\boldsymbol{\xi}$, of the fluid. Then, bearing in mind that the equations of motion are to be linearized:

$$\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} \simeq \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{3.1}$$

I now separate every scalar, dependent variable, say f, into its equilibrium value f_0 and a perturbation. I consider two kinds of perturbation: the Eulerian perturbation, namely the perturbation to f at a given position $\boldsymbol{r}$, which I denote by f', and the Lagrangian perturbation, which is the perturbation at a point following the motion and which I denote by δf. Thus

$$f(\boldsymbol{r}, t) = f_0(r) + f'(\boldsymbol{r}, t), \tag{3.2}$$

$$f(\boldsymbol{r} + \boldsymbol{\xi}, t) = f_0(r) + \delta f(\boldsymbol{r}, t). \tag{3.3}$$

The two perturbations are related by an equation which, after linearization with respect to perturbation quantities, becomes

$$\delta f = f' + \boldsymbol{\xi} \cdot \nabla f_0 \tag{3.4}$$

$$= f' + \boldsymbol{\xi} \cdot \boldsymbol{n} \frac{\mathrm{d} f_0}{\mathrm{d}r}, \tag{3.5}$$

where $\boldsymbol{n}$ is a unit vector in the outward radial direction. I will employ both Eulerian and Lagrangian perturbations at will, selecting whichever is the more convenient for my purposes.

Substituting either the form (3.2) or (3.3) into the equations of motion (1.2.1)–(1.2.5), subtracting the corresponding equations for the equilibrium state, and linearizing in perturbation quantities, yields

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p' - g_0 \boldsymbol{n} \rho' + \rho_0 \nabla \Phi', \tag{3.6}$$

$$\frac{\mathrm{D} f'}{\mathrm{D}t} = \frac{\partial f'}{\partial t} + \frac{\mathrm{d} f_0}{\mathrm{d}r} \boldsymbol{n} \cdot \boldsymbol{u} \tag{3.7}$$

$$= \frac{\mathrm{d}\delta f}{\mathrm{d}t}, \tag{3.8}$$

$$\nabla^2 \Phi' = -4\pi G \rho', \tag{3.9}$$

$$\delta\rho + \rho_0 \operatorname{div} \boldsymbol{\xi} = \rho' + \operatorname{div} \rho_0 \boldsymbol{\xi} = 0. \tag{3.10}$$

Notice that the partial derivative with respect to time is an Eulerian partial derivative, at fixed $\boldsymbol{r}$. The Lagrangian time derivative is defined to be the derivative following the motion, and is thus a full derivative when considered to be operating on a Lagrangian perturbation; the equivalence of eq. (3.7) and eq. (3.8) is therefore evident, and can be demonstrated formally by using eqs. (3.1) and (3.5) to expand the expression (3.8). Equations (3.10) were obtained by integrating the mass conservation equations (1.2.4) with respect to time, requiring that $\delta\rho = 0$ and therefore $\rho' = 0$ when $\boldsymbol{\xi} = 0$.

Equations (3.6)–(3.10) must be supplemented by the linearized perturbation to the adiabatic relation (1.2.5), which can be written

$$\delta p = c_0^2\, \delta\rho. \tag{3.11}$$

Except where it might cause confusion, I now simplify the notation by omitting the subscript zero from equilibrium quantities.

The oscillation equations (3.6)–(3.11) are homogeneous, linear equations for perturbations with time-independent coefficients. Likewise, the coefficients in the linearized boundary conditions, which I have not yet written, are also independent of time; and aside from an arbitrary amplitude-normalization condition that one must formally impose to determine the solution completely, the boundary conditions are also homogeneous. Consequently, the problem admits solutions that are separable in space and time, with exponential time dependence:

$$f'(r,t) = \mathrm{Re}[\tilde{f}'(r)\,\mathrm{e}^{-\mathrm{i}\omega t}], \tag{3.12}$$

where Re denotes real part, with similar expressions for δf and $\boldsymbol{\xi}$. The partial differential equation satisfied by the amplitude $\tilde{f}'(r)$ is essentially elliptic in nature, and admits nontrivial solutions satisfying the homogeneous boundary conditions on the surface enclosing the star only for specific eigenvalues of ω. The complete solution represents what is normally called a *mode* of oscillation. In general one would expect ω to be complex; it can be separated into its real and imaginary part:

$$\omega = \omega_{\mathrm{R}} + \mathrm{i}\omega_{\mathrm{I}}. \tag{3.13}$$

I call ω_{R} the frequency and ω_{I} the growth rate of the mode. Except at the beginning of appendix V, I will consider ω_{R} to be positive throughout these lectures, and, as will become apparent soon, I will be considering conditions under which $\omega_{\mathrm{I}} = 0$. Hence, ω will be real, but until that has been justified the reader is advised to consider the possibility that it is actually complex. I will simplify the notation yet further by dropping the tilde from the amplitude $\tilde{f}'(r)$, and, hopefully without causing confusion, use f', δf and $\boldsymbol{\xi}$ for both the time-dependent perturbations and the time-independent amplitude eigenfunctions.

4. Radial pulsations

Radial pulsations are spherically symmetrical oscillations, such that the velocity is in the radial direction. Because they are geometrically so simple, they have been studied in much greater detail than their aspherical counterparts; indeed, as will be seen later, the geometrical constraint even excludes an important class of motion, thereby contributing further to the relative simplicity of the discussion.

4.1. Linearized equations of motion

I introduce a dimensionless measure $\xi(r,t)$ of the radial component of the displacement $\boldsymbol{\xi}(r,t)$, defined such that with respect to spherical polar coordinates (r,θ,ϕ),

$$\boldsymbol{\xi} = (\xi, 0, 0)\,r. \tag{4.1.1}$$

Taking out the factor r is common practice in the literature, and I do so here too so that my discussion can be compared easily with others. But let me warn the reader now that, once again in keeping with quite common practice, I will not extract a factor r in the discussion of oscillations that are not spherically symmetric. I permit this incongruity, partly so that once again my discussion can be more easily related to much of what is already in the literature.

One of the simplifications afforded by spherical symmetry is the possibility of integrating the Poisson equation (3.9), thereby reducing the order of the differential system that remains to be solved. Thus, $\nabla\Phi' = -(Gm'/r^2)\boldsymbol{n}$, where $m' = 4\pi\int\rho' r^2\,\mathrm{d}r$. Actually it is more convenient to work with Lagrangian variables. The momentum equation then becomes

$$r\rho\ddot{\xi} = -\frac{\partial\delta p}{\partial r} + 4g\rho\xi, \tag{4.1.2}$$

and the equation of conservation of mass (3.10) is

$$\frac{\delta\rho}{\rho} + r^{-2}\frac{\partial}{\partial r}(r^3\xi) = 0, \tag{4.1.3}$$

where a dot denotes a (Lagrangian) time derivative. Equation (4.1.2) can be derived, after some manipulation, directly from eq. (3.6), by substituting $\nabla\Phi'$, replacing p' and ρ' with $\delta p - r\xi\,\mathrm{d}p/\mathrm{d}r$ and $\delta\rho - r\xi\,\mathrm{d}\rho/\mathrm{d}r$ and using eq. (4.1.3) and the hydrostatic equations (2.1)–(2.3), satisfied by the equilibrium state. It is actually much simpler, however, first to transform the spherically symmetrical form of eq. (3.6) to a Lagrangian coordinate system, so that m takes on the role of the independent variable and the radius $r(1+\xi)$ at fixed m is a dependent variable, and then transform back to the original (Eulerian) coordinate system after linearization.

I now eliminate δp from the momentum equation (4.1.2) using eq. (3.11), and use eq. (4.1.3) to eliminate $\delta\rho$, yielding

$$r\rho\ddot{\xi} + 4\frac{\mathrm{d}p}{\mathrm{d}r}\xi - \frac{\partial}{\partial r}\left[\gamma p\left(r\frac{\partial\xi}{\partial r} + 3\xi\right)\right] = 0. \tag{4.1.4}$$

Next, I introduce the separated form (3.12) for ξ, and drop the tilde from the amplitude ξ. I also rewrite the equation in self-adjoint form, which is obtained after multiplying it by the integrating factor r^3. The manipulations are straightforward, and lead to

$$\mathcal{L}\xi = 0, \tag{4.1.5}$$

where

$$\mathcal{L}\xi := \frac{\mathrm{d}}{\mathrm{d}r}\left(\gamma p r^4\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) + \left\{r^3\frac{\mathrm{d}}{\mathrm{d}r}[(3\gamma-4)p] + r^4\rho\omega^2\right\}\xi. \tag{4.1.6}$$

A result of using spherical polar coordinates is that the system is reduced to a set of ordinary differential equations. It is now necessary not only to apply boundary conditions at the surface $r = R$ of the star, but also to impose regularity conditions at the coordinate singularity $r = 0$. Thus the problem is transformed into an ordinary two-point boundary-value problem.

4.2. *Boundary conditions*

It is evident from its definition, eq. (4.1.6), that the operator $\mathcal{L}$ is singular at $r = 0$, since after expanding the first term and dividing by r^3 to ensure that at least one of the coefficients is nonzero at $r = 0$ without any becoming infinite, the coefficient of the highest derivative of ξ vanishes. The boundary condition that must be applied is therefore that which eliminates the singular solution. This is most readily obtained by expanding the solution as a power series in r:

$$\xi = r^a \sum_{k=0}^{\infty} A_k r^k, \tag{4.2.1}$$

with $A_0 \neq 0$. It is also necessary to expand the equilibrium quantities:

$$\begin{aligned} p &= p_0 - p_2 r^2 + \cdots, \qquad \rho = \rho_0 - \rho_2 r^2 + \cdots, \\ \gamma &= \gamma_0 - \gamma_2 r^2 + \cdots, \end{aligned} \tag{4.2.2}$$

the regularity conditions requiring, as explained in section 2, that there are no terms linear in r. The leading term of the first term of $\mathcal{L}\xi$ is thus $a(a+3)\gamma_0 p_0 A_0 r^{a+2}$, which cannot be balanced by the second term of $\mathcal{L}$ which is $O(r^{a+4})$ as $r \to 0$. Therefore, the indicial equation is $a(a+3) = 0$, and hence $a = -3$ or 0. The first value corresponds to the singular solution, which must be rejected. The second value yields the regular solution, which after inspecting higher terms in the expansion (4.2.2) can be seen to be of the form:

$$\xi \sim A_0 + A_2 r^2 + \cdots. \tag{4.2.3}$$

It follows that

$$\frac{d\xi}{dr} = 0 \quad \text{at } r = 0. \tag{4.2.4}$$

There is some diversity in the way the outer boundary condition is obtained, and also in where it is applied. One approach, which, as I will explain soon, is valid for modes with low frequency, is to ignore the details of the atmosphere and to approximate the outer layers of the star by a completely plane-parallel polytrope

with constant index $\mu > 0$. (The index μ has no relation with the mean molecular mass, introduced in section 1.1.) Then p and ρ vanish at the surface $r = R$ of the star. Indeed, $p \propto z^{\mu+1}$ and $\rho \propto z^{\mu}$, where z is the depth beneath the surface (see appendix I). The point $z = 0$ is a regular singular point of the operator $\mathcal{L}$ defined by eq. (4.1.6), which can be analyzed in precisely the same manner as the regular singular point at $r = 0$. The indicial equation associated with eq. (4.1.5) is $a(a+\mu) = 0$, from which follows once again that $a = 0$ for the regular solution. Unlike the situation at $r = 0$, however, the linear term in the series expansion does not vanish. Consequently $d\xi/dr$ does not vanish at the surface. The resulting boundary condition therefore relates ξ to its first derivative:

$$\gamma R \frac{d\xi}{dr} + \left(3\gamma - 4 - \frac{\omega^2}{\omega_0^2}\right)\xi = 0 \quad \text{at } r = R, \tag{4.2.5}$$

where R is the radius in the unperturbed stellar model and

$$\omega_0^2 = \frac{GM}{R^3} \tag{4.2.6}$$

defines a characteristic dynamical time scale ω_0^{-1} of the star. It should be pointed out that it is not really necessary to assume a plane-parallel polytrope. The regularity condition (4.2.5) follows for any stellar model at the surface of which p and ρ vanish in such a way that $d\ln p/d\ln\rho =: 1 + \mu^{-1}$ with $\mu \to \mu_s > 0$ as $r \to R$, which requires that $c \to 0$ as $r \to R$.

In reality, of course, the pressure and density do not vanish at the surface of the star, and therefore $r = R$ is not a singular point. However, one can argue that as one approaches the surface from below, p and ρ become very small compared with their values in the deep interior, and the solutions behave as though they are in the vicinity of a singular point. Thus, unless the true boundary condition were to select purely the solution that behaves like the singular solution (which it does not), the "singular" component of the actual solution would decay inwards to such an extent that for practical purposes the eigenfunction in the deep interior would be as if the singular component were entirely absent. Inspection of the expansion that leads to the condition (4.2.5) reveals that this argument is correct provided that in the region where the sound speed, c, deviates from polytropic behaviour and ceases to appear to be approaching zero at a finite value of r, $\omega^2 \ll c^2/H^2$ is satisfied, where H is the density or pressure-scale height. This quantifies what I meant by saying that the argument is valid for low frequencies. The significance of the inequality will become clear later. In any case, it is evident from this argument that if conditions are such that the boundary condition (4.2.5) is valid, it does not much matter precisely where in the surface layers it is applied. Typically, it is applied near the photosphere.

The boundary condition (4.2.5) is quite adequate for studying the lower-frequency radial pulsations of any star, even though it is perhaps not ideal (see section 4.8.2); in particular, it is an adequate condition for studying the classical variable stars such as the Cepheids and RR Lyrae stars, for which it is usually applied in the vicinity of the photosphere.

For studies of higher-frequency modes, such as those observed in the Sun, more care must be exercised. A common procedure in the solar case is to integrate the pulsation equation (4.1.5) through the atmosphere up to the base of the corona, which exerts a small pressure on the atmosphere. Because the corona is at a very high temperature, it is very rarefied; it has been argued that the absence of substantial inertia implies that the variation of the pressure it exerts on the chromosphere during the pulsation is negligible. Hence, one might think of applying the approximate condition

$$\delta p = 0 \quad \text{at } r = R, \tag{4.2.7}$$

and indeed this condition is sometimes used instead of condition (4.2.5). It seems to me that it would make more sense to argue that since the sound speed is so high the hydrostatic structure of the corona adjusts essentially instantaneously to low-frequency pulsations. Hence, the hydrostatic condition

$$4\pi r^2 p = g m_c, \tag{4.2.8}$$

where $g m_c$ is the weight of the corona, would be approximately satisfied at the moving interface between the chromosphere and the corona. The linearized perturbation equation obtained from this equation, assuming m_c to be constant, is

$$2\xi + \frac{\delta p}{p} = -2\xi \quad \text{at } r = R. \tag{4.2.9}$$

With the help of the adiabatic relation (3.11) and the continuity equation (4.1.3), this becomes

$$\gamma R \frac{\mathrm{d}\xi}{\mathrm{d}r} + (3\gamma - 4)\,\xi = 0 \quad \text{at } r = R. \tag{4.2.10}$$

Interestingly, this is formally the extreme low-frequency (static) limit of the boundary condition (4.2.5) obtained above. In particular, like condition (4.2.7) it is perfectly reflecting, as is the condition (4.2.4) at $r = 0$, and therefore permits the confinement of a mode within the star; it is intuitively obvious that, since there is no loss or gain of energy, ω is real in this case. A formal demonstration is presented in appendix II. It will be shown in section 4.6, however, that $\omega^2/\omega_0^2 \geqslant 3\gamma - 4$. Therefore it is inconsistent to ignore the frequency-dependent dynamical term compared with the geometrical term $3\gamma - 4$.

It is more realistic to solve the pulsation equations in the corona and to match the solution in the chromosphere to the causal coronal oscillation. This is straightforward if one assumes the corona to be isothermal, and leads to a condition like eq. (4.2.5). Another procedure that is sometimes adopted, is to assume the entire atmosphere to be isothermal and to apply the matching condition in the photosphere. In either case, it is necessary to study first the pulsations of an isothermal atmosphere.

4.2.1. Oscillations of an isothermal atmosphere

In the plane-parallel isothermal atmosphere under constant gravity, g, described in appendix I, eq. (4.1.5) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{-r/H}\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) + \frac{\omega^2}{c^2}\mathrm{e}^{-r/H}\,\xi = 0, \tag{4.2.11}$$

where $c^2 = \gamma g H$ and H is the scale height, both of which are constant, and, consistent with assuming g to be constant, I have ignored terms that are $\mathrm{O}(H/R)$. The solutions are exponential in r:

$$\xi \propto \exp(\tilde{\kappa} r), \tag{4.2.12}$$

where

$$\tilde{\kappa} = \frac{1}{2H}\left[1 \pm \left(1 - \frac{4\omega^2 H^2}{c^2}\right)^{1/2}\right]. \tag{4.2.13}$$

The causal solution is that obtained by selecting the negative sign; although ξ increases rapidly with height (i.e. with an e-fold height comparable with H), the energy density, which is proportional to $\rho\xi^2$, does not. (The energy density does increase rapidly with height for the other solution. Causal solutions are discussed more extensively in appendix V.)

If ω is real and less than ω_c, where

$$\omega_c = \frac{c}{2H}, \tag{4.2.14}$$

then $\tilde{\kappa}$ is real and the energy density decreases exponentially with height. The oscillation is said to be evanescent. In that case the atmosphere acts as a perfectly reflecting boundary for the oscillations in the interior of the star. Since the oscillations are adiabatic, there is no mechanism by which energy can be lost or gained, so ω is indeed real, as was assumed at the outset. If, on the other hand, $\mathrm{Re}(\omega) > \omega_c$, then $\tilde{\kappa}$ is complex, and the causal solution corresponds to a wave propagating upwards in the atmosphere. The energy density is now independent of height (for real ω) and energy is propagated away from the star. What would happen in reality under such circumstances is that the amplitude of the wave would become so large that energy would be dissipated by nonlinear processes such as shocks. The amplitude of the oscillation in the star would therefore decay with time, unless it were maintained either by nonlinear interactions with other motions or by nonadiabatic processes which here are being ignored. Of course, to be consistent it is therefore necessary to take the fact into account that ω is now really complex, and since energy is being lost one expects $\omega_I < 0$. Indeed, it can then be demonstrated that if the motion in the star is everywhere truly adiabatic, then $\omega_I < 0$ when $\omega_R > \omega_c$; the argument is a simple extension of that which I use in section 4.5 and appendix II to show that ω is real when $\omega_R < \omega_c$.

Since ω is not real, the energy density of the oscillations in the atmosphere is not precisely independent of height. The higher in the atmosphere $r > R$, the earlier the time at which the outwardly propagating waves had previously crossed the surface $r = R$. Since the amplitude at fixed r is declining with time, it follows that the energy density increases with height for fixed t, at a rate that is proportional to ω_I. This is demonstrated formally in appendix V.

The quantity ω_c is the critical frequency below which waves cannot propagate vertically in an isothermal atmosphere under constant gravity. It was first determined by Lamb (1908), and is sometimes named after him. I will not adopt that practice in these lectures, however, because I wish ω_c not to be confused with another frequency, associated with nonradial oscillations, which is defined by eq. (5.5.3) and is also called the Lamb frequency. Radial oscillations of the atmosphere are essentially acoustic waves. If the atmosphere were uniform, waves would propagate with speed c, but, as we have already seen, propagation is inhibited by stratification. The force restoring a simple localized adiabatic compression or rarefaction is provided by the gradient of the pressure perturbation. If the characteristic spatial scale λ of the perturbation is much less than the scale height H of the equilibrium stratification, that stratification can be ignored as a first approximation, and the wave behaves essentially as though it were propagating in a uniform medium; the gradient of the pressure perturbation is greatest between regions of compression and rarefaction, and has such a sign as to oppose the density perturbation. If, on the other hand, $\lambda \gg H$, the gradient of the pressure perturbation is dominated by the basic stratification, and is greatest where the density is greatest, irrespective of whether it is in a state of compression or rarefaction. The result is essentially a bodily shift of the perturbation, with no local tendency to restore the original equilibrium: wave propagation no longer occurs. Indeed, since the bulk of the atmosphere is contained within a characteristic height of order H, one can

envisage it to be essentially finite; a high-frequency oscillation imposed on the base would cause sound waves to propagate upwards, but a low-frequency oscillation simply lifts the entire atmosphere up and down in phase with the base. The transition between the two kinds of behaviour evidently occurs when λ is comparable with H. Taking $\lambda = k^{-1}$, where k is the wave number of a wave satisfying the acoustic dispersion relation $kc = \omega$, this condition becomes $c/\omega \simeq H$, which to within a factor two is the condition $\omega = \omega_c$. Note that if the perfect-gas law, eq. (1.1.4), applies, $\omega_c \propto T^{-1/2}$.

The solution (4.2.12) can be used to derive a boundary condition for the oscillation eigenfunctions in the body of the star. It is obtained by matching the eigenfunctions with the causal solution in the atmosphere in such a way that ξ and δp are continuous. The latter condition is obtained by integrating the momentum equation (4.1.2) across the interface at $r = R + \xi$, separating the isothermal atmosphere from the interior. If one ignores any discontinuity of γ that may exist at that interface, it follows from eqs. (4.1.3) and (3.11) that the continuity of ξ and δp implies continuity of $\mathrm{d}\xi/\mathrm{d}r$. Hence, one may write

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} - \tilde{\kappa}(\omega)\xi = 0 \quad \text{at } r = R, \tag{4.2.15}$$

where of course ξ and $\mathrm{d}\xi/\mathrm{d}r$ are evaluated at $r = R_-$ but $\tilde{\kappa}$ is evaluated (from eq. (4.2.13) with the negative sign) at $r = R_+$. Here $r = R$ would be either the photosphere or the base of the corona if there is one. Provided $\omega^2 \ll \omega_c^2$, the square root in eq. (4.2.13) may be expanded about unity. Keeping only the leading-order surviving term then reduces the condition (4.2.15) to

$$\gamma R\frac{\mathrm{d}\xi}{\mathrm{d}r} - \frac{\omega^2\gamma RH}{c^2}\xi \simeq 0 \quad \text{at } r = R. \tag{4.2.16}$$

Since $c^2 = \gamma g H$, this is similar to condition (4.2.5), except that the term $3\gamma - 4$ is missing. That term arises from spherical geometry. Evidently, since we expect to find oscillations with $\omega = \mathrm{O}(\omega_0)$, the neglect of spherical geometry in the atmosphere is formally invalid. In practice, however, both the geometrical terms and the dynamical terms are numerically unimportant when $\omega \simeq \omega_0$, so $\mathrm{d}\xi/\mathrm{d}r = 0$ at $r = R$ is a good approximation, even though that condition would not select the regular solution if c^2 were to vanish at the surface. A better condition that takes into account both the dynamics and the geometry of the region beyond $r = R$ is obtained in section 4.8.2.

Note that even though ω_c is much smaller in the corona than it is in the atmosphere beneath, the corona transmits only a small proportion of the energy, provided that the transition region between the chromosphere and the corona is much thinner than an oscillation wavelength and can therefore be considered discontinuous. The acoustic energy flux is proportional to $c\rho\xi^2$. The pressure p and displacement ξ are continuous across the interface, from which it can easily be shown that the transmission coefficient is approximately four times the ratio of the atmospheric and the coronal sound speed, which in the case of the Sun is about 20%.

Finally, it is important to realize that in any stellar atmosphere there is a region in which the motion is not adiabatic. The results of this adiabatic study must therefore be regarded as no more than an introductory guide to the dynamics. It is worth pointing out in this connection that when the oscillation is not adiabatic beneath the photosphere, some care must be taken in selecting the appropriate solution. This is discussed in appendix V, where the adiabatic response of an atmosphere to growing or decaying oscillatory forces from below is studied.

4.3. Orthogonality of eigenfunctions

Equation (4.1.4), subject to the boundary conditions (4.2.4) and (4.2.5) or (4.2.15), admits a sequence of eigenfunctions ξ_n with eigenvalues ω_n^2. The system is similar to the Sturm–Liouville problem, but it is not identical since the outer boundary condition depends on the eigenvalue. Nevertheless, the system shares many of its properties with the Sturm–Liouville system, particularly because boundary terms arising from integration by parts are usually negligible.

Let us multiply eq. (4.1.4) for ξ_n by ξ_k and integrate over the interval $(0, R)$ of r. On integrating the first term by parts one obtains

$$\left[\gamma p r^4 \xi_k \frac{\mathrm{d}\xi_n}{\mathrm{d}r}\right]_0^R - \int_0^R \gamma p r^4 \frac{\mathrm{d}\xi_k}{\mathrm{d}r}\frac{\mathrm{d}\xi_n}{\mathrm{d}r}\,\mathrm{d}r + \int_0^R \left\{r^3 \frac{\mathrm{d}}{\mathrm{d}r}\left[(3\gamma - 4)p\right] + r^4 \rho \omega_n^2\right\} \xi_k \xi_n \,\mathrm{d}r = 0. \tag{4.3.1}$$

It is common practice to ignore the integrated term; it is zero at $r = 0$ and one would expect it to be very small at $r = R$ since p very nearly vanishes. (The ratio of the pressure in the solar photosphere to that at the median radius in the Sun is about 10^{-10}. However, the argument is not complete unless it is demonstrated that $\xi_k\,\mathrm{d}\xi_n/\mathrm{d}r$ is not 10^{10} times greater in the surface layers than it is in the interior of the star. Once again, as we will see later, that condition is satisfied provided $\omega^2 \ll c^2/H^2$.) Thus, if one similarly multiplies eq. (4.1.4) for ξ_k by ξ_n, integrates and subtracts the result from eq. (4.3.1), one obtains

$$\left(\omega_n^2 - \omega_k^2\right) \int_0^R r^4 \rho \xi_k \xi_n \,\mathrm{d}r = 0, \tag{4.3.2}$$

so, provided $\omega_k^2 \neq \omega_n^2$, ξ_k and ξ_n are orthogonal over $(0, R)$ with respect to the weight function $r^4\rho$. Therefore the eigenfunctions appear to form a convenient basis for expanding the solutions of problems that differ in only a minor way from the simple problem considered here.

In artificial stellar models in which p/ρ strictly vanishes at the surface, the integrated term in eq. (4.3.1) is always zero and the arguments of Sturm–Liouville theory carry over to this problem. There is an infinite sequence of orthogonal eigenfunctions, which form a complete set of functions on the interval $(0, R)$, satisfying $\mathrm{d}\xi/\mathrm{d}r = 0$ at $r = 0$; the modes can be ordered according to the value of ω_n, and the zeros of consecutive eigenfunctions ξ_n of that ordering interlace. The eigenfunction ξ_1, associated with the lowest eigenvalue, has no internal node. Consequently, the displacement eigenfunction $r\xi_n$ has n nodes, if one includes the zero at $r = 0$.

In realistic stellar models the integrated term in eq. (4.3.1) is not zero, and is no longer small compared to the other terms in the equation once ω^2 is comparable to the value of ω_c^2 at $r = R$. In that case there are only a finite number of modes. The eigenfunctions are almost orthogonal when $\omega^2 \ll \omega_c^2$; this would be true even if the integrated term cannot be ignored, provided the approximate boundary condition (4.2.16) is valid. But when ω^2 is comparable to ω_c^2, orthogonality is lost.

4.4. Some nomenclature

When the modes are arranged in order of increasing frequency, the lowest frequency corresponding to $n = 1$, the label n is called the *order* of the mode. The mode of lowest order is called the *fundamental* (preferably, *fundamental radial mode* if there is any danger of confusing it with the nonradial f mode discussed in section 5) and the higher modes are called *overtones*; in keeping with standard musical practice, the mode of order $n + 1$ is the nth overtone.

This nomenclature is not universally adopted. It is common to call the overtones of stellar pulsations harmonics, even though the frequencies do not form a harmonic sequence. Moreover, contrary to standard musical nomenclature, the first overtone is not called the second harmonic, as it should have been had it been a harmonic, but instead it is called the first harmonic. Despite this inconsistency, people in the field seem to be able to communicate without undue difficulty. Nevertheless, I shall refrain from misusing the term harmonic.

4.5. Variational principle

Equation (4.1.5) subject to the boundary conditions (4.2.4) and (4.2.15), constitutes a self-adjoint problem, and therefore results from a simple variational principle. Consider the functional

$$\tilde{\omega}^2(\xi) := \frac{K(\xi, \xi^*) - B(\xi, \xi^*)}{I(\xi, \xi^*)}, \tag{4.5.1}$$

where the asterisk denotes complex conjugate and

$$K(\xi,\eta) := \int_0^R \left\{ \gamma p r^4 \frac{\mathrm{d}\xi}{\mathrm{d}r}\frac{\mathrm{d}\eta}{\mathrm{d}r} - r^3 \frac{\mathrm{d}}{\mathrm{d}r}\left[(3\gamma-4)p\right]\xi\eta \right\} \mathrm{d}r, \tag{4.5.2}$$

$$I(\xi,\eta) := \int_0^R \rho r^4 \xi\eta \, \mathrm{d}r, \tag{4.5.3}$$

$$\mathcal{B}(\xi,\eta) := \gamma p r^4 \bar{\kappa} \eta\xi \,\big|_{r=R} \tag{4.5.4}$$

are defined for all twice differentiable functions ξ and η that satisfy the boundary conditions (4.2.4) and (4.2.15). Equation (4.5.1) was obtained from eq. (4.3.1) by setting $\xi_n = \xi$, $\xi_k = \eta$ and $\omega_n^2 = \widetilde{\omega}^2$. It is straightforward to show in the usual way that $\widetilde{\omega}^2$ is stationary with respect to variations $\delta\xi$ to ξ satisfying the boundary conditions, when ξ is an eigenfunction ξ_n of the problem (4.1.4), (4.2.4), (4.2.15), and the stationary values of $\widetilde{\omega}^2$ are the corresponding eigenvalues ω_n^2.

One can also demonstrate the reality of the eigenvalues. If $\omega_n^2 \ll \omega_c^2$, so that the condition (4.2.15) can be replaced by eq. (4.2.16), it follows that

$$\left[I(\xi,\xi^*) + \rho r^4 H \xi\xi^*\big|_{r=R}\right]\omega^2 = K(\xi,\xi^*). \tag{4.5.5}$$

All the terms except possibly ω^2, in this equation are real, from which follows that ω^2 must be real. The coefficients in eq. (4.1.5) and the boundary conditions (4.2.4) and (4.2.16) are therefore real, from which follows that one can choose ξ to be real. This argument is generalized in appendix II to the case when ω^2/ω_c^2 is not very small, and the boundary condition (4.2.15) must be used instead of eq. (4.2.16); the result is that if ω is real it must be smaller than ω_c.

4.6. Lower bound to the pulsation frequency

A simple lower bound can easily be derived from eq. (4.5.1) provided the boundary term $\mathcal{B}$ can be ignored, γ is considered to be constant and provided the mean density $\bar{\rho}(r)$ of that portion of the star within the sphere of constant r decreases outwards (a condition that always appears to be satisfied in stable theoretical stellar models). Since the first term of the integrand of $K(\xi,\xi)$ is positive definite (see eq. (4.5.2)), it follows that

$$\omega^2 \geqslant (3\gamma-4)\frac{\int_0^R r^3 (Gm\rho/r^2)\xi^2\,\mathrm{d}r}{\int_0^R \rho r^4 \xi^2\,\mathrm{d}r} = \frac{4\pi}{3}(3\gamma-4)G\bar{\rho}(R)\frac{\int_0^R [\bar{\rho}(r)/\bar{\rho}(R)]\xi^2\,\mathrm{d}u}{\int_0^R \xi^2\,\mathrm{d}u}, \tag{4.6.1}$$

where

$$\bar{\rho}(r) = \frac{3m}{4\pi r^3} \tag{4.6.2}$$

and

$$\mathrm{d}u = \rho r^4\,\mathrm{d}r. \tag{4.6.3}$$

In deriving eq. (4.6.1) from eq. (4.5.1) the hydrostatic equations (2.1)–(2.3) of the equilibrium state have been used. Since it has been assumed that $\bar{\rho}(r)/\bar{\rho}(R) \geqslant 1$, it follows from the inequality (4.6.1) that

$$\omega^2 \geqslant (3\gamma-4)\frac{GM}{R^3} = (3\gamma-4)\omega_0^2. \tag{4.6.4}$$

It has already been pointed out that ω_0 is a typical dynamical frequency. Now we see, since $3\gamma - 4 = 1$ if $\gamma = \frac{5}{3}$, that it gives a lower bound of ω. This lower bound cannot be achieved in practice, since it would require the density to be uniform, which would be convectively unstable. (If ρ = const., eq. (4.1.4) is satisfied by ξ = const., and the condition (4.6.1) is satisfied with equality.) A more stringent bound, applicable to solar-type stars, is discussed by Christensen-Dalsgaard et al. (1983).

Notice that if $\gamma > \frac{4}{3}$, $\omega^2 > 0$ and the star is dynamically stable. By way of comparison it is interesting to note that the fundamental frequency ω_1 of a polytrope with index 3 is about $3\omega_0$; the values for theoretical models of the present Sun are about $2.6\omega_0$.

4.7. Elementary discussion of dynamical stability

Consider the Roche model of a stellar envelope, discussed in appendix III. For the purpose of calculating the gravitational acceleration, g, it is assumed in constructing that model that all the mass of the star is concentrated at a point at the centre, which simplifies the analysis tremendously. I use the model simply as a guide, and do not assume that γ is constant (even though I used the perfect-gas law with μ constant to obtain the structure (A3.6)). Substituting the solution (A3.6) into the adiabatic pulsation equation (4.1.4) yields

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} + \frac{r}{c_0^2 r_0}\left[\omega^2 - (3\gamma-4)\frac{GM}{r^3}\right]\xi = 0, \tag{4.7.1}$$

where c_0 is the sound speed at some reference value r_0 of r. Assume $\omega^2 \neq 0$ and consider the fundamental mode, for which ξ has no node. Select a normalization such that $\xi > 0$ and recall that $\mathrm{d}\xi/\mathrm{d}r = 0$ at $r = 0$ (condition (4.2.4)). Either $\mathrm{d}\xi/\mathrm{d}r < 0$ or $\mathrm{d}\xi/\mathrm{d}r \geqslant 0$ at $r = R$. Suppose the former: then $\mathrm{d}^2\xi/\mathrm{d}r^2 < 0$ somewhere, and there, according to eq. (4.5.1), $S := \omega^2 - (3\gamma - 4)GM/r^3 > 0$. Suppose now that $\mathrm{d}\xi/\mathrm{d}r \geqslant 0$ at $r = R$; I adopt the outer boundary condition (4.2.5) and deduce immediately that $S \geqslant 0$ at $r = R$. Thus $S \geqslant 0$ somewhere, whatever the functional form of the fundamental eigenfunction ξ. Consequently, if the star is dynamically unstable ($\omega^2 < 0$) to radial perturbations, it must be the case that $\gamma < \frac{4}{3}$ somewhere.

What I have just presented is far from a general proof, because it depends on the structure of the Roche model and it rests on the premise (which is correct for realistic stellar models, but which I have not proved) that the fundamental eigenfunction has no node. The result that γ must be less than $\frac{4}{3}$ somewhere in a star that is unstable to radial adiabatic perturbations is, however, a general result. It has been proved, e.g., by F.J. Dyson (unpublished), using an energy argument.

4.8. The Liouville–Green expansion of high-order modes

When $n \gg 1$, $\omega \gg \omega_0$ and the eigenfunctions ξ of eq. (4.1.4) oscillate rapidly with r. In that case one can utilize the Liouville–Green (JWKB) approximation. The principle of the approximation is to write the eigenfunction as the product of a rapidly oscillating sinusoidal function with an appropriate phase and a slowly varying amplitude, A. Thus one might set ξ equal to

$$\mathrm{Re}\left[A \exp\left(\mathrm{i}\lambda \int \psi\, \mathrm{d}r\right)\right], \tag{4.8.1}$$

where $\lambda := \omega/\omega_0$ is large and A and ψ vary on the same length scale as the equilibrium state. One then substitutes the form (4.8.1) into eq. (4.1.4) and essentially equates to zero the coefficient of each power of λ.

4.8.1. Reduction to standard form

Before proceeding I will cast eq. (4.1.4) into a form that contains no first derivative of the dependent variable, in the expectation that the leading-order terms will then give a more accurate representation of the solution. This is suggested by the damped-oscillator equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\kappa\frac{\mathrm{d}y}{\mathrm{d}x} + K^2y = 0, \quad 0 \leqslant x \leqslant 1 \tag{4.8.2}$$

where here κ and K are constants, and K is large. One could expand y in powers of κ/K about the solution of $\mathrm{d}^2y/\mathrm{d}x^2 + K^2y = 0$, but this requires some work

(see appendix IV). However, if instead the first-derivative term is eliminated by replacing y with $\eta = y\exp(\kappa x)$, the new equation

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} + (K^2 - \kappa^2)\eta = 0 \tag{4.8.3}$$

is actually solved exactly by a function of the form of eq. (4.8.1), provided one has the good sense to include the κ^2 term at the outset. Evidently, when K and κ vary slowly with x, one would still expect to reap the benefits of such a transformation. Accordingly, I set

$$\xi = r^{-2}\rho^{-1/2}c^{-1}\Xi, \tag{4.8.4}$$

which transforms eq. (4.1.5) into

$$\frac{\mathrm{d}^2\Xi}{\mathrm{d}r^2} + K^2\Xi = 0, \tag{4.8.5}$$

where now

$$K^2 = \frac{\omega^2 - \omega_\mathrm{c}^2}{c^2} \tag{4.8.6}$$

is the square of a characteristic wave number (and is not constant), and ω_c is a critical acoustic frequency which is given by

$$\omega_\mathrm{c} = \frac{c}{2\mathcal{H}}\left[1 + 2\mathcal{H}' + \frac{4(3\gamma-4)\mathcal{H}^2}{\gamma r H_p} + \frac{12\mathcal{H}^2}{rH_\gamma}\right]^{1/2}, \tag{4.8.7}$$

the prime denoting differentiation with respect to the argument r; also I require various scale heights:

$$\begin{aligned} &H^{-1} := -\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}, \qquad H_p^{-1} := -\frac{\mathrm{d}\ln p}{\mathrm{d}r}, \qquad H_\gamma^{-1} := -\frac{\mathrm{d}\ln\gamma}{\mathrm{d}r},\\ &\mathcal{H}^{-1} := H_p^{-1} + H_\gamma^{-1} - 4r^{-1},\\ &H_c^{-1} := -\frac{\mathrm{d}\ln c}{\mathrm{d}r} = \frac{1}{2}(H_\gamma^{-1} + H_p^{-1} - H^{-1}), \end{aligned} \tag{4.8.8}$$

the first of which will not be needed until later. The density and pressure-scale heights are defined, as usual, with a minus sign, to make H_p positive everywhere and $\mathcal{H}$ positive throughout most of the star; I include a minus sign in the definitions of H_γ and H_c for consistency, and will also do so in future for all other scale heights.

The quantity ω_c defined here is a generalization of Lamb's critical frequency, which for a plane-parallel, isothermal atmosphere under constant gravity in which γ is constant was given by eq. (4.2.14). It clearly reduces to Lamb's value under those restrictive conditions. Note, furthermore, that ω_c is relatively large near the surface of the star, where scale heights are small; it is evident from making either the isothermal or the polytropic approximation that $\omega_c^2 \propto \gamma g/H$, the constant of proportionality, which depends on the stratification of the star, being of order unity. Furthermore, ω_c diverges as $r \to 0$.

4.8.2. Radial oscillations of the isothermal atmosphere revisited

Having obtained the standard form of the radial-pulsation equation, one can now take geometry into account in the discussion of the oscillations of an isothermal atmosphere ($r \geqslant R$) with constant γ. Using the equilibrium state (A1.14), (A1.15) of appendix I, one can immediately establish that

$$\frac{d^2\Xi}{dr^2} + \frac{\omega^2 - \omega_{c0}^2}{c^2}\Xi \simeq -\frac{r-R}{RH^2}\Xi, \quad r \geqslant R, \tag{4.8.9}$$

where c is constant and

$$\omega_{c0} \simeq \frac{c}{2H}\left[1 - \frac{4(2-\gamma)H}{\gamma R}\right] \tag{4.8.10}$$

is the value of ω_c at $r = R$. As far as conditions at the base of the atmosphere $r = R$ are concerned, it is only in the lower few scale heights that the structure of the solution is important. Therefore the right-hand side of eq. (4.8.9) is small, and can be used as the basis of a perturbation expansion. (Even though eq. (4.8.9) is essentially Airy's equation, whose solutions are well documented, it is most convenient to develop the expansion ab initio for this small range of the independent variable.) Thus one can set $\Xi = \tilde{\Xi}_0 + HR^{-1}\tilde{\Xi}_1 + \cdots$, where here $\tilde{\Xi}_i$ are functions of r satisfying

$$\tilde{\Xi}_0'' - \kappa^2\tilde{\Xi}_0 = 0, \tag{4.8.11}$$

$$\tilde{\Xi}_1 - \kappa^2\tilde{\Xi}_1 = (r-R)H^{-3}\tilde{\Xi}_0, \ldots, \tag{4.8.12}$$

where $\kappa^2 = (\omega_{c0}^2 - \omega^2)/c^2$. The causal solution (which is that which is not forced from above) for the case $\kappa^2 > 0$ is

$$\tilde{\Xi}_0 = \Xi_0\, e^{-\kappa r}, \tag{4.8.13}$$

$$\tilde{\Xi}_1 = \frac{r-R}{H^3\kappa^2}\Xi_0\,[1 + \kappa(r-R)]\,e^{-\kappa r}, \ldots, \tag{4.8.14}$$

where Ξ_0 is a constant.

Since ξ and $d\xi/dr$ are continuous at $r = R$, at least when γ is continuous, a boundary condition at $r = R$ to be applied to the solution for $r \leqslant R$ can be obtained by eliminating Ξ_0 between Ξ and Ξ' and expressing the result in terms of ξ using the transformation (4.8.4):

$$\frac{d\xi}{dr} + [\kappa - \tfrac{1}{2}H^{-1} + R^{-1}(2 - \tfrac{1}{4}H^{-2}\kappa^{-2})]\xi = 0 \quad \text{at } r = R, \tag{4.8.15}$$

which replaces condition (4.2.15). It is evidently preferable to either eq. (4.2.5), (4.2.10) or (4.2.15). For modes with $\omega^2 \ll \omega_c^2$, κ can be expanded about $c^{-1}\omega_{c0}$, reducing the boundary condition to

$$\gamma R\frac{d\xi}{dr} + \left[3\gamma - 4 - \frac{\omega^2}{\omega_0^2}\right]\xi \simeq 0 \quad \text{at } r = R, \tag{4.8.16}$$

which is identical to condition (4.2.5).

4.8.3. The JWKB approximation

We are now in a position to proceed with the approximation of the solution in the interior of the star. The idea is to set

$$\Xi = A\exp\left(i\lambda\int\psi\, dr\right), \tag{4.8.17}$$

where $R\psi$ is of order unity, substitute into eq. (4.8.5), yielding

$$-\lambda^2\psi^2 A + i\lambda(2\psi A' + \psi' A) + A'' + K^2 A = 0, \tag{4.8.18}$$

and equate powers of λ (cf. appendix IV). However, I am not going to adhere strictly to the rules. I recall not only the advantage of retaining the small term κ^2 in eq. (4.8.3), but also that ω_c becomes large near the surface of the star and diverges at the centre. I therefore consider ω_c to be possibly $O(\lambda^2)$. I will solve only the leading-order equations for A and ψ, so it will not be necessary to expand these functions in inverse powers of λ, as was done in appendix IV. The equations are (cf. eqs. (A4.8) and (A4.10)):

$$\psi^2 = \frac{\omega^2 - \omega_c^2}{\lambda^2 c^2} \tag{4.8.19}$$

and

$$2\psi\frac{dA}{dr} + \frac{d\psi}{dr}A = 0, \tag{4.8.20}$$

from which

$$\lambda\psi = \pm c^{-1}(\omega^2 - \omega_c^2)^{1/2} \tag{4.8.21}$$

and

$$A = \Xi_0(\lambda\psi)^{-1/2} = \Xi_0 c^{1/2}(\omega^2 - \omega_c^2)^{-1/4}, \tag{4.8.22}$$

where Ξ_0 is again a constant.

Stopping at this order is the JWKB approximation. I emphasize again, however, that by first casting the pulsation equation into the standard form (4.8.5), the approximation at this order is likely to be considerably more accurate than the outcome of expanding the raw equation (4.1.4) to even the next order. Moreover, it requires substantially less effort.

4.8.4. Form of the solution: wave trapping

Where $K^2 > 0$ ($\omega_c < \omega$), ψ is real and the solution (4.8.17) is oscillatory: the upper and lower signs in eq. (4.8.21) yield waves propagating outwards and inwards, respectively. A linear combination of two such waves with the same amplitude produces a standing wave:

$$\xi \sim \Xi_0(r^4\rho c)^{-1/2}(\omega^2 - \omega_c^2)^{-1/2} \sin\left[\int (\omega^2 - \omega_c^2)^{1/2}\frac{\mathrm{d}r}{c}\right]. \tag{4.8.23}$$

Where $\omega_c > \omega > 0$, ψ is imaginary and the solution is evanescent. This is always the case near the centre of the star, and is also so near the surface, provided ω does not exceed the value of the critical acoustic frequency in the stellar atmosphere. The mode is then said to be trapped between the two evanescent regions. In the analysis that follows I will assume that the mode is evanescent in a finite region immediately beneath $r = R$.

4.8.5. Bridging the transition: eigenvalue equation

The transition levels between the evanescent layers and the region of propagation, $r = r_1$ and $r = r_2$, are turning points of eq. (4.8.5). There, $\omega_c = \omega$ and A would be infinite if eq. (4.8.22) were valid, which of course strictly speaking it is not: as $\omega_c \to \omega$, eq. (4.8.22) implies that $|A'/A| \to \infty$, which contradicts the ordering of the terms assumed in the expansion of eq. (4.8.18).

The transition at the turning point can be obtained from Olver's (1974) comparison method. The principle is to approximate eq. (4.8.5) by another equation with the same mathematical properties and which can be integrated more readily through the turning point. Since in general the turning point is a simple zero of K^2 (i.e. $\mathrm{d}K^2/\mathrm{d}r \neq 0$ at the point where $K^2 = 0$), the appropriate comparison equation is Airy's equation, which I write in the form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + xy = 0. \tag{4.8.24}$$

There is an integral representation of the solutions of this equation, valid for all x. Therefore one can connect appropriate asymptotic approximations in the evanescent and oscillatory regions, valid for $x \to -\infty$ and $x \to +\infty$, either by evaluating the integral by the method of stationary phase or by consulting any standard text on special functions (e.g. Abramowitz and Stegun (1964)). For $x \to +\infty$, the solution $\mathrm{Ai}(-x)$ that decays to zero from above as $x \to -\infty$, is given by

$$\mathrm{Ai}(-x) \sim \pi^{-1/2}x^{-1/4}\sin(\tfrac{2}{3}x^{3/2} + \tfrac{1}{4}\pi) + \mathrm{O}(x^{-5/4}); \tag{4.8.25}$$

the solution for $x \to -\infty$ is

$$\mathrm{Ai}(-x) \sim \tfrac{1}{2}\pi^{-1/2}(-x)^{-1/4}\exp[-\tfrac{2}{3}(-x)^{3/2}]\,[1 + \mathrm{O}(x^{-1})]. \tag{4.8.26}$$

The other solution is $\mathrm{Bi}(-x)$, whose asymptotic behaviour far from the turning point is

$$\mathrm{Bi}(-x) \sim \pi^{-1/2}x^{-1/4}\cos(\tfrac{2}{3}x^{3/2} + \tfrac{1}{4}\pi) \quad \text{for } x \to \infty, \tag{4.8.27}$$

$$\mathrm{Bi}(-x) \sim \pi^{-1/2}(-x)^{-1/4}\exp[\tfrac{2}{3}(-x)^{3/2}] \quad \text{for } x \to -\infty. \tag{4.8.28}$$

The reduction of eq. (4.8.5) is accomplished by the Liouville transformation

$$x = \mathrm{sgn}(K^2)\left|\tfrac{3}{2}\int_{r_1}^{r} K\,\mathrm{d}r\right|^{2/3}, \tag{4.8.29}$$

$$\Psi = |x|^{-1/4}|K|^{1/2}\Xi, \tag{4.8.30}$$

leading to

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + x\Psi = h(|x^{1/2}K^{-1}|)\Psi, \tag{4.8.31}$$

where

$$h[s(r)] := s^{1/2}\frac{\mathrm{d}^2 s^{-1/2}}{\mathrm{d}r^2}. \tag{4.8.32}$$

It has been proved that an asymptotic expansion of the solution of eq. (4.8.31) can then be obtained by regarding the right-hand side as a small perturbation to the Airy equation, provided, of course, h does not diverge (Olver 1956, 1974, Langer 1959). I will retain only the leading terms $\mathrm{Ai}(-x)$ and $\mathrm{Bi}(-x)$ of that expansion.

To determine the combination of Ai and Bi that represents the solution to the pulsation problem, we must first consider the expansions in the evanescent regions near $r = 0$ and $r = R$, where the boundary conditions are applied. First consider the solution for $r \to 0$. Unfortunately h diverges as $r \to 0$, and the asymptotic expansion is invalidated. This should be no surprise since we know that $r = 0$ is a singular point of eq. (4.8.5). The existence of a singular point near the turning point of the equation renders its mathematical structure different from that of the Airy equation (4.8.24); to be correct we should use a more complicated comparison equation, such as Bessel's equation. However, in an attempt to keep the analysis simple, one might consider retaining the Airy-function approximation in the hope that the influence of the singular point is appreciable only in the tail of the evanescent decay inwards from $r = r_1$, notwithstanding the fact that $r_1 \sim \sqrt{2}\omega^{-1}c(0) \to 0$ as $\omega \to \infty$, and does not have a severe effect on the solution in the propagating region. Thus, we evaluate the asymptotic representations (4.8.26) and (4.8.28) for Ai and Bi as $r \to 0$, obtaining $\xi \propto r^a$, where $a = -\frac{3}{2} + \sqrt{2} = -0.086$ for Ai and $a = -\frac{3}{2} - \sqrt{2} = -2.91$ for Bi. Although these values are not identical to the roots 0 and -3 of the indicial equation for the singular point, they are quite close; it is evident that Bi must be rejected and that Ai corresponds to the regular solution (4.2.3). The asymptotic expansion (4.8.25) of that solution far from $r = r_1$ in the region of propagation, $r_1 < r < r_2$, is

$$\Xi \sim \Xi_1 K^{-1/2} \sin\left(\int_{r_1}^{r} K(s)\,\mathrm{d}s + \frac{\pi}{4}\right), \tag{4.8.33}$$

where Ξ_1 is a constant.

Near the surface I assume that K^2 has a single zero, at $r = r_2$. The eigenfunction ξ is therefore evanescent all the way from $r = r_2$ to the surface. If the solution is

$$\Psi \propto \mathrm{Ai}(-x) - \epsilon \mathrm{Bi}(-x) \tag{4.8.34}$$

for some constant ϵ, with x defined by eq. (4.8.29) with $r_i = r_2$, then for $r > r_2$ and far from r_2

$$\Xi \sim \tilde{\Xi}_2 |K|^{-1/2}\left[\frac{1}{2}\exp\left(-\int_{r_2}^{r}|K(s)|\,\mathrm{d}s\right) - \epsilon\exp\left(\int_{r_2}^{r}|K(s)|\,\mathrm{d}s\right)\right], \tag{4.8.35}$$

where $\tilde{\Xi}_2$ is also a constant. Application of the boundary condition (4.8.15), which in terms of Ξ is $\mathrm{d}\Xi/\mathrm{d}r + (\kappa - \frac{1}{4}R^{-1}H^{-2}\kappa^{-2})\Xi = 0$ at $r = R$, determines the

constant ϵ. If the geometrical term is ignored, the condition becomes

$$\frac{\mathrm{d}\Xi}{\mathrm{d}r} + \kappa\Xi = 0 \quad \text{at } r = R, \tag{4.8.36}$$

and hence

$$\epsilon = \frac{1}{2}\frac{\kappa - |K(R)|}{\kappa + |K(R)|}\,\mathrm{e}^{-2d}, \tag{4.8.37}$$

where

$$d = \int_{r_2}^{R} |K(r)|\,\mathrm{d}r. \tag{4.8.38}$$

Note that $|K(R)|$ is interpreted as the limit of $|K(r)|$ as $r \to R$ from below; κ is the limit of $|K(r)|$ as $r \to R$ from above. The two are not necessarily equal, because c and ω_c could be regarded as being discontinuous at the interface $r = R$. The solution in the region of propagation, $r_1 < r < r_2$, far from the turning points is thus

$$\Xi \sim \Xi_2 K^{-1/2} \sin\left(\int_{r}^{r_2} K(s)\,\mathrm{d}s + \frac{\pi}{4} - \tan^{-1}\epsilon\right), \tag{4.8.39}$$

where $\Xi_2 = (1 + \epsilon^2)\tilde{\Xi}_2 \simeq \tilde{\Xi}_2$.

The two expressions (4.8.33) and (4.8.39) must be identical, and are also equivalent to eq. (4.8.23) with appropriate limits of integration. Therefore $\Xi_2 = \pm\Xi_1$, and formally

$$\int_{r_1}^{r_2} K\,\mathrm{d}r = (n - \tfrac{1}{2})\pi + \tan^{-1}\epsilon, \quad n = 1, 2, \ldots. \tag{4.8.40}$$

Equation (4.8.40) determines the eigenvalues ω_n of ω. Since the asymptotic expansion was developed subject to the assumption that

$$\lambda = \frac{\omega}{\omega_0} = \mathrm{O}\left(\omega\int_0^R \frac{\mathrm{d}r}{c}\right) = \mathrm{O}\left(\int_{r_1}^{r_2} K\,\mathrm{d}r\right) \gg 1, \tag{4.8.41}$$

it follows that eq. (4.8.40) is actually valid asymptotically for large n.

It is important to note that the depth of the evanescent region increases as ω decreases: for the plane-parallel polytropic envelope with index μ discussed in appendix I, e.g., $\omega_c^2 = \frac{1}{4}(\mu - 1)\gamma g z^{-1}$, and therefore $R - r_2 = \frac{1}{4}(\mu - 1)\gamma g \omega^{-2}$. Consequently, for relatively low-order modes the details of the upper boundary condition, and therefore the structure of the upper atmosphere, are not very important. This is reflected in the factor e^{-2d} in eq. (4.8.37) for ϵ, which when $\omega^2 \ll \omega_c^2$ is approximately $\{\frac{1}{2}e[(\mu+1)/(\mu-1)]\omega/\omega_c\}^a$, where $a = 2\sqrt{(\mu^2-1)}$ and ω_c is evaluated in the atmosphere for $r > R$. A more precise evaluation of ϵ for this model is presented in appendix VI. As far as acoustic oscillations are concerned, the outer layers of the Sun, e.g., can be approximated by a polytrope with index 3 (see section 6 and appendix VIII). Therefore the influence of the atmosphere is seen to be a rapidly varying function of ω, and contributes approximately unity to the integral in eq. (4.8.40) when $\omega \approx \omega_c$. In the Sun, $\omega \simeq \omega_c$ when $n \simeq 35$. Thus, the relative contribution that the atmosphere makes to the highest-frequency modes is $\epsilon/n\pi$, which is about 1%, and is not insignificant. It is a straightforward matter to verify that this quantity also is an estimate of the ratio of B to K in eq. (4.5.1), as indeed it must be.

Finally, it is instructive to simplify the asymptotic expression (4.8.40) for the frequencies. The critical acoustic frequency, ω_c, is comparable with ω only very near the surface of the star and, in the form given by eq. (4.8.7), near the centre. (The natural definition of ω_c depends on the initial choice of dependent and independent variables; another expression, based on the use of the Lagrangian pressure perturbation as dependent variable, is given by eq. (5.4.9). Unlike expression (4.8.7), that expression does not diverge at $r = 0$. Thus it can be ignored almost everywhere.) Near the centre, c is approximately constant and $\omega_c \sim -c/r$, as can be seen by expanding expression (4.8.7) about the origin. The contribution from ω_c near the centre can therefore be estimated from

$$\begin{aligned}\int_{r_1}^{r} (\omega^2-\omega_c^2)^{1/2}c^{-1}\,dr &\sim \int_{c/\omega}^{r}(\omega^2-c^2/r^2)^{1/2}c^{-1}\,dr\\ &= \frac{r\omega}{c} - \cos^{-1}\left(\frac{c}{r\omega}\right)\\ &\sim \omega\int_0^r \frac{dr}{c} - \frac{\pi}{2},\end{aligned} \tag{4.8.42}$$

the asymptotic limit being valid for large $\omega r/c$ and $\omega H_c/c$. The contribution from ω_c in the surface layers can be estimated by using once again the polytropic model of appendix I. Now I ignore the atmosphere, because I have already estimated its influence. Consider ω_c to be insignificant below, say, $z = Z$. Then the contribution to the integral in eq. (4.8.40) above $z = Z$ is approximately

$$T(Z) = \frac{\omega z_0}{c_0}\int^{Z}\left[1-\frac{1}{4}(\mu^2-1)\frac{c_0^2}{\omega^2 z_0 z}\right]^{1/2}\left(\frac{z_0}{z}\right)^{1/2}\frac{dz}{z_0}, \tag{4.8.43}$$

the lower limit of the integration being where the integrand vanishes, which is

taken to be the reference depth z_0. Evaluation of the integral is straightforward:

$$\begin{aligned}T(Z) &= \sqrt{\mu^2-1}\left[\frac{2\omega}{c_0}\left(\frac{Zz_0}{\mu^2-1}\right)^{1/2} - \frac{\pi}{2}\right]\\ &= \int_0^Z \frac{dz}{c} - \frac{\pi}{2}\sqrt{\mu^2-1}.\end{aligned} \tag{4.8.44}$$

It follows immediately from eq. (4.8.40) that when ϵ is ignored, the frequency is given by

$$\omega \sim (n + \tfrac{1}{2}\sqrt{\mu^2-1})\omega_0, \tag{4.8.45}$$

where here

$$\omega_0 = \left(\pi\int_0^R \frac{dr}{c}\right)^{-1} \tag{4.8.46}$$

is a new characteristic dynamical frequency (different, yet similar in magnitude, to the quantity with the same name given by eq. (4.2.6)) defined in terms of the characteristic acoustic travel time $\int c^{-1}\,dr$ from the centre to the surface of the star. Equation (4.8.44) resembles the equation for the resonant frequencies of an organ pipe, with a constant shift $\frac{1}{2}\sqrt{\mu^2-1}$ resulting from effective phase jumps at the two reflecting ends. It exhibits the fact that, since the influence of the stratification of the star is significant only near $r = 0$ and $r = R$, it can be represented simply as an effective change of boundary conditions. It is interesting to observe that, on taking account of the contribution from the Airy function, the lower turning point contributes $-\frac{1}{4}\pi$ and the upper turning point $\frac{1}{2}(\frac{1}{2}-\sqrt{\mu^2-1})\pi$ to the effective phase jump. Thus, eq. (4.8.45) is comparable to the exact result (A6.28) with $\delta = 0$ for a complete plane-parallel polytrope on a rigid base (at which there is no phase jump), which suggests that the accuracy of eq. (4.8.45) might be improved by replacing $\sqrt{\mu^2-1}$ by μ. A similar replacement is suggested in section 5.8 for nonradial modes.

5. Nonradial oscillations about a spherically symmetric state

The term "nonradial" in this context is a misnomer: it refers to motion that is not purely radial, and not necessarily to motion that has no radial component. Indeed, in a spherically symmetrical star, there is no mode with a nonzero frequency that has no radial component of the displacement $\boldsymbol{\xi}$.

5.1. The equations of motion

Because the equilibrium state is spherically symmetrical, with respect to spherical polar coordinates (r, θ, ϕ) eqs. (3.1)–(3.11) admit separable solutions of the form

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \mathrm{Re}\left[\left(\xi(r)P_l^m, \frac{\eta(r)}{L}\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}, \frac{\mathrm{i}m\eta(r)}{L\sin\theta}P_l^m\right)\mathrm{e}^{\mathrm{i}m\phi - \mathrm{i}\omega t}\right], \tag{5.1.1}$$

and

$$\psi(\boldsymbol{r}, t) = \mathrm{Re}[\psi(r)P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi-\mathrm{i}\omega t}] \tag{5.1.2}$$

for any scalar Eulerian or Lagrangian perturbation ψ, where $P_l^m(\cos\theta)$ is the associated Legendre function of the first kind of degree l and order m, and

$$L = [l(l+1)]^{1/2}. \tag{5.1.3}$$

Note that in the case of scalar perturbations I use the same symbol to denote both the complete perturbation and its r-dependent amplitude. Note also that ξ is now the amplitude of the vertical component of the displacement, and is not scaled with r^{-1}, as it was in eq. (4.1.1) for radial pulsations. With this decomposition, eqs. (3.1), (3.6)–(3.8) and (3.10) can easily be reduced to the following two differential equations in ξ and p', η having been eliminated using eq. (3.10) and the horizontal divergence of eq. (3.6), and the density perturbation having been eliminated using eq. (3.11):

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi) - \frac{g}{c^2}\xi + \left(1 - \frac{L^2c^2}{\omega^2r^2}\right)\frac{p'}{\rho c^2} = -\frac{L^2}{\omega^2r^2}\Phi', \tag{5.1.4}$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} + \frac{g}{c^2}p' + (N^2 - \omega^2)\rho\xi = \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \tag{5.1.5}$$

(e.g. Unno et al. (1979)), where N is the buoyancy frequency, given by

$$N^2 = g\left(\frac{1}{H} - \frac{g}{c^2}\right) \tag{5.1.6}$$

and H is the density-scale height defined by eq. (4.8.8). The reduced Poisson equation determining the amplitude of the Eulerian perturbation to the gravitational potential is

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) - \frac{L^2}{r^2}\Phi' = -4\pi G\left(\frac{p'}{c^2} + \frac{N^2\rho}{g}\xi\right) \tag{5.1.7}$$

$$= -4\pi G\left(\frac{\delta p}{c^2} + \frac{\rho}{H}\xi\right). \tag{5.1.8}$$

I find it convenient to express the equations in terms of the Lagrangian pressure fluctuation δp. This is accomplished with the transformation (3.5). After some straightforward manipulation, eqs. (5.1.4) and (5.1.5) become

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} + \left(\frac{2}{r} - \frac{L^2g}{\omega^2r^2}\right)\xi + \left(1 - \frac{L^2c^2}{\omega^2r^2}\right)\frac{\delta p}{\rho c^2} = -\frac{L^2}{\omega^2r^2}\Phi', \tag{5.1.9}$$

$$\frac{\mathrm{d}\delta p}{\mathrm{d}r} + \frac{L^2g}{\omega^2r^2}\delta p - \frac{g\rho f}{r}\xi = \rho\left(\frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \frac{L^2g}{\omega^2r^2}\Phi'\right), \tag{5.1.10}$$

where the discriminant f is given by

$$f = \frac{\omega^2r}{g} + 2 + \frac{r}{H_g} - \frac{L^2g}{\omega^2r}, \tag{5.1.11}$$

H_g being the scale height of the gravitational acceleration, g, defined with the sign convention of eq. (4.8.8).

Equations (5.1.8)–(5.1.11), with appropriate boundary conditions at $r = 0$ and $r = R$, constitute an eigenvalue problem for a general adiabatic mode whose displacement eigenfunction has the form (5.1.1), (5.1.2). The degree l of the associated Legendre function is called the *degree* of the mode, and the order m I will refer to as the *azimuthal order* of the mode. Radial modes have $l = 0$, and indeed in that case, with the appropriate solution of eq. (5.1.8) and after some tedious manipulations, eqs. (5.1.9) and (5.1.10) can be reduced to eqs. (4.1.5) and (4.1.6) for the dimensionless displacement. For each value of l (and m) the eigenvalues form a discrete sequence, and will again be labelled with an integer n called the *order* of the mode. As we will see later, except when $l = 0$ there are two distinct types of modes, one with high frequency and the other with low frequency. The latter are sometimes assigned negative values of n.

5.2. Boundary conditions

As in the case of radial oscillations, the centre of the star is a regular singular point. The regularity conditions are

$$\Phi' = \mathrm{O}(r^l) \tag{5.2.1}$$

and

$$\delta p = \mathrm{O}(r^l), \tag{5.2.2}$$

or equivalently

$$\xi = \mathrm{O}(r^\alpha), \quad \alpha = l - 1 \quad \text{if } l \geqslant 1, \quad \alpha = 1 \quad \text{if } l = 0, \tag{5.2.3}$$

as $r \to 0$. Thus

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}r} - \frac{l}{r}\Phi' \to 0 \quad \text{as } r \to 0, \tag{5.2.4}$$

with a similar equation for δp, or

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} - \frac{\alpha}{r}\xi \to 0 \quad \text{as } r \to 0. \tag{5.2.5}$$

The boundary condition for Φ' at the surface $r = R$ is determined by matching Φ' and its derivative with a causal vacuum field. Thus, $\Phi' \propto r^{-l-1}$ for $r \geqslant R$, and hence

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \frac{l+1}{r}\Phi' = 0 \quad \text{at } r = R. \tag{5.2.6}$$

The dynamical condition is obtained by matching with the causal solution in the region $r \geqslant R$. If that region is assumed to be isothermal, and the plane-parallel approximation is made, the equation of motion for $r \geqslant R$ (which is obtained by eliminating δp using eqs. (5.1.9) and (5.1.10) with $\Phi' = 0$, neglecting the perturbed gravitational potential being at present an unjustified assumption, to which I will return later) is given approximately by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{-r/H}\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) + \left[\frac{\omega^2}{c^2} - k^2\left(1 - \frac{N^2}{\omega^2}\right)\right]\mathrm{e}^{-r/H}\xi = 0, \tag{5.2.7}$$

where $k = L/R$, which reduces to eq. (4.2.11) when $k = 0$. It is immediately evident from the analysis of radial pulsations that, at least when ρ and γ are continuous, the appropriate boundary condition is

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} + \tilde{\kappa}(\omega)\xi = 0 \quad \text{at } r = R \tag{5.2.8}$$

(cf. appendix VI), where $\tilde{\kappa}(\omega)$ is given by eq. (4.2.13) with ω^2 replaced by $\omega^2 - k^2c^2(1 - N^2/\omega^2)$. Care must be taken to choose the correct square root (appendix V). It is straightforward to incorporate the geometrical terms, as was described for radial oscillations in section 4.8, and the influence of the perturbation of the gravitational potential.

It is important to notice that neither the governing differential equations nor the boundary conditions explicitly depend on m. Therefore the eigenfrequencies $\omega = \omega_{nl}$ are also independent of m.

5.3. Variational principle

An integral relation for the frequency can be obtained directly from the basic linearized equations of motion, as in section 4.5. Substituting $-\mathrm{i}\omega\boldsymbol{\xi}$ for $\boldsymbol{u}$ in eq. (3.6), taking the scalar product with the complex conjugate, $\boldsymbol{\xi}^*$, of $\boldsymbol{\xi}$, solving eq. (3.9) for Φ', eliminating p' and ρ' using eqs. (3.11) and (3.10) and integrating over the unperturbed volume, $\mathcal{V}$, of the star yields

$$\omega^2 = \frac{K(\boldsymbol{\xi},\boldsymbol{\xi}^*) - \mathcal{B}(\boldsymbol{\xi},\boldsymbol{\xi}^*)}{I(\boldsymbol{\xi},\boldsymbol{\xi}^*)}, \tag{5.3.1}$$

where

$$\begin{aligned} K(\boldsymbol{\xi},\boldsymbol{\eta}) := &\int_{\mathcal{V}} [\gamma p \operatorname{div}\boldsymbol{\xi}\operatorname{div}\boldsymbol{\eta} + \boldsymbol{\xi}\cdot\nabla p \operatorname{div}\boldsymbol{\eta} + \boldsymbol{\eta}\cdot\nabla p\operatorname{div}\boldsymbol{\xi} \\ &+ \rho^{-1}(\boldsymbol{\xi}\cdot\nabla\rho)(\boldsymbol{\eta}\cdot\nabla p)]\,\mathrm{d}V \\ &- G\iint_{\mathcal{V}} \frac{\operatorname{div}[\rho(\boldsymbol{r})\boldsymbol{\xi}(\boldsymbol{r})]\operatorname{div}'[\rho(\boldsymbol{r}')\boldsymbol{\eta}(\boldsymbol{r}')]}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V\,\mathrm{d}V', \end{aligned} \tag{5.3.2}$$

$$I(\boldsymbol{\xi},\boldsymbol{\eta}) := \int_{\mathcal{V}} \rho\boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}V \tag{5.3.3}$$

and

$$\mathcal{B}(\boldsymbol{\xi},\boldsymbol{\eta}) := \int_{\mathcal{S}} \rho(c^2\operatorname{div}\boldsymbol{\xi} - \boldsymbol{g}\cdot\boldsymbol{\xi} - \Phi')\boldsymbol{\eta}\cdot\mathrm{d}\boldsymbol{S}, \tag{5.3.4}$$

where

$$\Phi'(\boldsymbol{r}) = -G\int_{\mathcal{V}} \frac{\operatorname{div}'[\rho(\boldsymbol{r}')\boldsymbol{\xi}(\boldsymbol{r}')]}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V', \tag{5.3.5}$$

$\mathcal{S}$ is the unperturbed surface of the star and div$'$ means the divergence with respect to the primed independent variable. This is the Eulerian analogue of eq. (4.5.1), but of course is valid for both radial and nonradial oscillations.

Suppose now for simplicity that the oscillations are evanescent in the outer layers of the star and that the surface $\mathcal{S}$ is taken to be sufficiently high in the atmosphere that the boundary term $\mathcal{B}$ can be ignored. It is immediately obvious from the symmetry of the integrals (5.3.2) and (5.3.3) that eq. (5.3.1) constitutes a variational principle for all functions $\boldsymbol{\xi}$ that do not cause $\mathcal{B}$ to be significantly large (e.g. Chandrasekhar (1964)). (The integral $K(\boldsymbol{\xi},\boldsymbol{\eta})$ is symmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ since in a spherically symmetrical star ∇p and $\nabla\rho$ are parallel.) It can also be shown that subject to suitable boundary conditions, eq. (5.3.1) is a variational principle when $\mathcal{B}$ is included, but I refrain from discussing that here, partly because the boundary conditions have not been discussed. By the same argument as was used in section 4.3, orthogonality of the eigenfunctions can be established. It follows also from the symmetry of $\mathcal{K}$ and $\mathcal{I}$ that ω^2 is real, at least when $\mathcal{B}$ is negligible; it is perhaps worth recording the physical tautology that, as is the case for radial pulsations, ω^2 is real for adiabatic motion whenever the boundary conditions are perfectly reflecting.

5.4. The Cowling approximation: reduction to standard form

By differentiating eq. (5.1.10) with respect to r, using eq. (5.1.9) to eliminate $\mathrm{d}\xi/\mathrm{d}r$ and then using eq. (5.1.10) to eliminate ξ, an equation relating δp to Φ' is obtained:

$$\frac{\mathrm{d}^2\delta p}{\mathrm{d}r^2} + \mathcal{H}^{-1}\frac{\mathrm{d}\delta p}{\mathrm{d}r} + \left[\frac{1}{c^2}\left(\omega^2 + \frac{g}{h}\right) - \frac{L^2}{r^2}\left(1 - \frac{\mathcal{N}^2}{\omega^2}\right)\right]\delta p = -\rho F, \quad (5.4.1)$$

where

$$\rho F = -\left(\frac{\mathrm{d}}{\mathrm{d}r} + \mathcal{H}^{-1} - \frac{L^2 g}{\omega^2 r^2}\right)\left(\rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \frac{L^2 g\rho}{\omega^2 r^2}\Phi'\right) + \frac{L^2 g\rho f}{\omega^2 r^3}\Phi', \quad (5.4.2)$$

$$h^{-1} = H_g^{-1} + 2r^{-1} \quad (5.4.3)$$

is the scale height of g/r^2,

$$\mathcal{H}^{-1} = H^{-1} + H_f^{-1} + h^{-1} + r^{-1}, \quad (5.4.4)$$

where H_f is the scale height of f, and

$$\mathcal{N}^2 = g\left(\frac{1}{\mathcal{H}} - \frac{g}{c^2} - \frac{2}{h}\right). \quad (5.4.5)$$

The difference between $\mathcal{N}^2$ and N^2 is the outcome solely of spherical geometry and self-gravity of the equilibrium state: in the limit of a plane-parallel envelope under constant gravitational acceleration $\mathcal{N}^2$ reduces to N^2. However, the scale height $\mathcal{H}$ is different from that defined by eq. (4.8.8) in the discussion of radial oscillations.

Cowling (1941) showed that, except for modes of low degree l with a numerically small order n, the perturbation Φ' to the gravitational potential has a relatively minor effect on the modes. Although Φ' must be included in accurate numerical computations of all but the high-degree modes, it has little influence on the basic dynamics (except for modes with $l = 0$ and $l = 1$, but I will not discuss this here), and consequently I will ignore it. Thus I set $F = 0$, and eq. (5.4.1) reduces to a single second-order differential equation for δp.

It will be convenient to reduce eq. (5.4.1) to the standard form of section 4.8.1. This is accomplished by the transformation

$$\delta p = \left(\frac{g\rho f}{r^3}\right)^{1/2}\Psi =: u\Psi, \quad (5.4.6)$$

resulting in the equation

$$\Psi'' + K^2\Psi = 0, \quad (5.4.7)$$

where as before the primes denote differentiation with respect to the argument (here r), and now

$$K^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{\mathcal{N}^2}{\omega^2}\right). \quad (5.4.8)$$

The critical acoustic frequency, ω_c, sometimes called the acoustic cutoff frequency, is defined by

$$\omega_c^2 = \frac{c^2}{4\mathcal{H}^2}(1 - 2\mathcal{H}') - \frac{g}{h}. \quad (5.4.9)$$

It is different from the definition (4.8.7). That is because here the dependent variable Ψ is based on the pressure perturbation, δp, whereas the radial oscillations were discussed in terms of the displacement, ξ. Since these variables do not have a constant ratio, their points of inflexion must be at different locations, and therefore the critical frequency (when $l = 0$) at those points must be different. Note that $\mathcal{H}$ is the scale height of the factor u defined in eq. (5.4.6).

The notation I have used was chosen to make the equations look similar to those that have already been developed by Deubner and Gough (1984), who did not take the spherical geometry fully into account. Equations (5.4.5) and (5.4.9) reduce to the corresponding equations of Deubner and Gough if $h^{-1} \to 0$ and $\mathcal{H} \to H$, and indeed approximate them well everywhere except very close to the centre of the star. I will call the limits of ω_c^2 and $\mathcal{N}^2$ as $h^{-1} \to 0$ the planar values.

5.5. Mode classification

Waves can propagate where $K^2 > 0$ and are evanescent where $K^2 < 0$. Because of the way in which K^2 depends on the two parameters characterizing the mode,

ω and l, it is more convenient to discuss the trapping in terms of the (positive) critical frequencies $\omega_\pm$ at which $K^2 = 0$. One can rewrite eq. (5.4.8) as

$$\omega^2 c^2 K^2 = (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2), \tag{5.5.1}$$

where

$$\omega_\pm^2 = \tfrac{1}{2}(S_l^2 + \omega_c^2) \pm [\tfrac{1}{4}(S_l^2 + \omega_c^2)^2 - \mathcal{N}^2 S_l^2]^{1/2}. \tag{5.5.2}$$

The quantity

$$S_l = \frac{Lc}{r} \tag{5.5.3}$$

is usually called the Lamb frequency.

The frequencies ω_+ and ω_-, computed at various values of l using the planar values of ω_c and $\mathcal{N}$, are plotted against r in fig. 2 for a theoretical model of the Sun. Note that the major spherical factor, in S_l^2, is correctly incorporated. The other spherical corrections are significant only very close to the centre of the Sun and could not easily be included in the figure because they depend, through $\mathcal{H}$, on ω. Except near the centre of the star, where ω_c is approximately constant and $\mathcal{N} = O(r)$, the frequencies ω_c and $\mathcal{N}$ are comparable in the radiative regions. (Except in the superadiabatic boundary layer immediately beneath the photosphere, $N^2 < 0$ and is very small throughout the convection zone; therefore $\mathcal{N}^2$ is small there too.) Well beneath the photosphere S_l^2 is substantially greater than ω_c^2 and $\mathcal{N}^2$, and therefore $\omega_+ \simeq S_l$ and $\omega_- \simeq \mathcal{N}$. In the atmosphere S_l^2 is typically much less than ω_c^2 and $\mathcal{N}^2$ (unless $l \gtrsim 2000$), and $\omega_+ \simeq \omega_c$ and $\omega_- \simeq 2L(H/R)N$.

A wave can propagate where $\omega > \omega_+$ or $\omega < \omega_-$ and is evanescent where $\omega_- < \omega < \omega_+$. High-frequency waves are determined mainly by the behaviour of ω_+, and hence depend predominantly on c and ω_c. They are a nonradial generalization of the spherically symmetrical acoustic oscillations discussed in section 4, and have been named *p modes* by Cowling (1941), because pressure perturbations provide the major contribution to the restoring force. Low-frequency waves are controlled mainly by the buoyancy frequency, N: Cowling called them *g modes* because gravity, through buoyancy, is the major contributor to the dynamics. They are standing internal-gravity waves.

Included in fig. 2 are several thin horizontal lines, representing the (constant) frequencies, ω, of various modes. They are drawn continuous in the regions of propagation and dashed where the modes are evanescent. On the whole, p modes are confined to an outer region of the star beneath the photosphere, the trapping region becoming shallower the higher the value of l. At frequencies near 5 mHz it is also possible for p modes to be trapped in the chromosphere. Provided $l \ll 2000$,

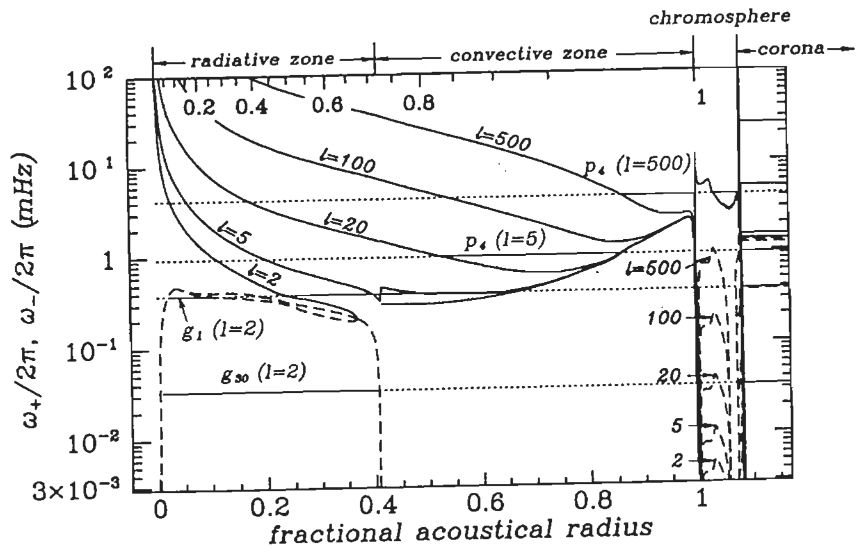


Fig. 2. Propagation diagram for a model of the Sun. The corona is represented by a plane-parallel isothermal atmosphere at temperature 1.5×10^6 K. Solid curves represent $\omega_+/2\pi$ and dashed curves $\omega_-/2\pi$, in the regions where the critical frequencies $\omega_\pm$ are real. Propagation at any frequency is possible where $\omega_\pm$ are complex. The lower abscissa is the acoustical radius, τ, measured in units of the acoustical radius, T, of the photosphere; the upper abscissa is the geometrical radius, r/R. The curves $\omega_\pm$ are for $l = 2, 5, 20, 100$ and 500. In all cases $\omega_\pm$ are increasing functions of l at fixed τ, which permits the identification of the curves in regions where they are not labelled explicitly: in the interior the ω_- curves for $l \geqslant 5$ are essentially indistinguishable, as are the ω_- curves for $l \geqslant 20$ in the corona and all four ω_+ curves in the chromosphere, where $\omega_+ \simeq \omega_c$. Notice the dip in the low-degree ω_+ curves centred at $\tau/T \simeq 0.83$, corresponding to $r/R \simeq 0.98$; it is due to the second ionization of helium, as is evident from fig. 1. The thin horizontal lines represent normal modes; they are continuous in zones of propagation and dashed in evanescent regions. The lowest-frequency mode is a pure g mode. Its amplitude is substantial only in the radiative interior; since it can propagate in the corona, some energy leakage from the Sun is possible. The next mode is $g_1(l = 2)$, which has the character of a p mode in its outer zone of propagation. The third mode is $p_4(l = 5)$, which is a simple p mode confined to a single region of propagation. The highest-frequency mode is $p_4(l = 500)$; this mode resonates in the subphotospheric region of propagation and in the chromosphere, and therefore has substantial amplitude in both regions.

the chromospheric eigenfunctions are almost independent of l, since the characteristic horizontal scale of variation, $\pi l^{-1} R$, is much greater than the vertical extent of the trapping region and locally all modes resemble radial modes. I will discuss later what happens when $l \gtrsim 2000$.

Gravity modes can be trapped either beneath the convection zone or in the atmosphere. Formally, a single mode can exist in both regions; however, the evanescent decay is so large in the turbulent convection zone that the extremely weak coupling between the atmosphere and the interior is likely to be destroyed by the turbulence. Thus, for practical purposes the interior modes and the atmospheric modes can be regarded as being distinct.

It is also important to have a formal mathematical classification. An unambiguous scheme has not been devised, except when the Cowling approximation ($\Phi' = 0$) is applied. For simple models of a star in which, e.g., ω_-^2 increases and ω_+^2 decreases monotonically with r for each l, the p modes and the g modes separate into two distinct groups. The p modes can be arranged in order of increasing ω and numbered with the order n, starting at $n = 1$ for the lowest-frequency mode and with n also being the number of nodes in ξ excluding the zero at $r = 0$ when $l \neq 0$. Similarly, when $l \neq 0$ the g modes can be arranged in a sequence of decreasing eigenfrequency, with n once again counting the nodes in ξ. The two sequences are separated by a mode with no node, whose frequency lies between those of the g modes and the p modes. This mode can exist only when $l \neq 0$. Cowling called it the f *mode*, for *fundamental* gravity mode.

The modes appear to form a well-ordered sequence even when the star is not simple. The reason is that only one condition of the form of eq. (5.2.8) relating $d\xi/dr$ to ξ, is applied at the boundary $r = R$, say. Solutions ξ of eqs. (5.1.9) and (5.1.10) with $\Phi' = 0$ satisfying the inner boundary condition (5.2.5) form a class of functions that vary continuously with ω^2, and pass through eigenfunctions ξ_n whenever condition (5.2.8) is satisfied. There might be an ambiguity in the ordering only if two modes could coincide, which basically would be possible only if an appropriate average of K^2 weighted predominantly over the region of propagation, were stationary with respect to ω^2; this is not the case since in that region K^2 is dominated by either ω^2/c^2 or $L^2\mathcal{N}^2/\omega^2 r^2$. Therefore it appears that the eigenfunctions are well separated. Consequently, if one now considered a realistic stellar model, to be obtained from the simple model by continuous deformation, since the modes are continuous functionals of the equilibrium state and cannot cross, the ordering is preserved. Therefore, in principle the order n of the mode computed with $\Phi' = 0$ is a well-defined quantity. I must emphasize that what I have said is a description and not a proof, since I have not actually proved that the modes cannot cross.

The order n does not necessarily count the nodes in ξ. As the equilibrium state is considered to deform, the eigenfunction ξ can develop convolutions, crossing the axis and generating new nodes in pairs. However, Scuflaire (1974) and Osaki (1975) have devised a classification scheme based on a discussion by Eckart (1960), which counts nodes algebraically according to the behaviour of p' or δp. My discussion here is similar, although not identical, to theirs. Equations (5.1.9) and (5.1.10) in the Cowling approximation can be rewritten:

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\psi\xi) = -\left(1 - \frac{S_l^2}{\omega^2}\right)\frac{r^2\psi}{\rho c^2}\delta p, \tag{5.5.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(\psi^{-1}\,\delta p) = \frac{g\rho f}{r\psi}\xi, \tag{5.5.5}$$

where

$$\psi = \exp\left(-\frac{L^2}{\omega^2}\int\frac{g}{r^2}\,\mathrm{d}r\right). \tag{5.5.6}$$

Let us concentrate our attention on nonradial ($l \neq 0$) p modes. If $f > 0$, which is normally the case in the region where p-mode eigenfunctions are oscillatory, then according to eq. (5.5.5) $\psi^{-1}\,\delta p$ has a minimum at a node of ξ at which $d\xi/dr > 0$ and a maximum at a node where $d\xi/dr < 0$. Moreover, since $\omega^2 > S_l^2$ in the oscillatory region, it follows from eq. (5.5.4) that the zeros of δp interlace with those of ξ, $r^2\psi\xi$ having maxima at zeros of δp at which $d\delta p/dr > 0$. Figure 3 illustrates schematically some simple examples of phase diagrams, in which δp is plotted against ξ as r varies from 0 to R. Illustrated in fig. 3a is a pure p mode of order unity. When r is small and eqs. (5.2.2) and (5.2.3) approximate the eigenfunctions, $\delta p/\xi > 0$; thus if the eigenfunction is normalized such that $\xi > 0$ as $r \to 0$, the phase plot starts in the first quadrant. Near the surface $\xi \neq 0$, and it is easy to see that $\delta p/\xi$ is again positive, so the plot must end in either the first or the third quadrant. When the equilibrium model is deformed continuously to a state appropriate to fig. 3b, the beginning and end of the phase plot must have remained in their appropriate quadrants throughout the deformation. Since, according to eqs. (5.5.4) and (5.5.5), ξ and δp cannot vanish simultaneously at a regular point (since ψ is everywhere positive), the phase path cannot cross the origin, and consequently the number of crossings of the δp-axis in the positive sense, counted algebraically, is preserved. A simple g mode of order 4, for which $f < 0$, is illustrated in fig. 3c. It has four crossings of the δp-axis in the negative sense. Note that in fig. 3b the first crossing is in the negative sense. Here the mode behaves as though it were a g mode.

The discrimination between p- and g-mode character is made by the sign of the factor f in eq. (5.5.5). Where f is small, $\psi^{-1}\,\delta p$ hardly varies, and in a simple stellar model ξ has no node: the mode is the f mode illustrated in fig. 3d. Of course, in a real star the f mode can have nodes in ξ, but the number of p nodes and g nodes cancel.

In view of this discussion it is sometimes convenient to assign negative integers n to the g modes. The f mode is designated $n = 0$. Then ω_n is a strictly increasing

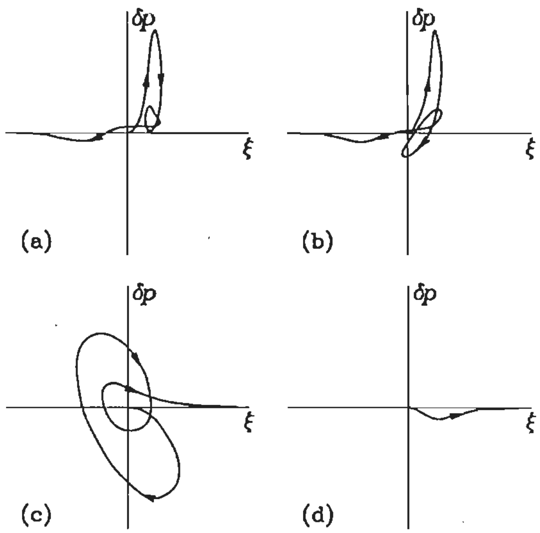


Fig. 3. Schematic phase diagrams. Plotted is δp against ξ, characterizing an eigenfunction as r varies from 0 to R, the arrows indicating the direction of increasing r. Case (a) is a pure p mode of order unity, (b) is also a p mode of order unity, but with a g-mode-like node near the centre of the star, (c) is a pure g mode of order 4 and (d) is a pure f mode.

function of n at constant l. When I refer to a g mode of order 3, say, what I really mean is therefore a mode of order -3.

Now a remark on the classification of the radial modes. If l is not constrained to be an integer, then the fundamental ($n = 1$) nonradial p-mode "eigenfrequency" tends continuously to the lowest radial eigenfrequency as $l \to 0$ (Vandakurov 1967a). For this reason I assigned order unity in section 4 to the fundamental radial mode.

Finally, I must emphasize that once the perturbation Φ' to the gravitational potential is included, the basis of this simple classification scheme collapses. When l is large, Φ' has little influence, and in practice a scheme of the kind of that discussed here probably suffices. But when l is small, some modification is necessary.

5.6. The f mode

If one approximates the equilibrium state locally in the vicinity of $r = r_0$, say, as part of a plane-parallel envelope under constant gravitational acceleration, eqs. (5.1.9)–(5.1.11) in the Cowling approximation reduce to

$$\frac{d\xi}{dr} - \frac{g_0 k^2}{\omega^2}\xi + \left(1 - \frac{k^2c^2}{\omega^2}\right)\frac{\delta p}{\rho c^2} = 0, \tag{5.6.1}$$

$$\frac{d\delta p}{dr} + \frac{g_0 k^2}{\omega^2}\delta p - r_0^{-1} f g_0 \rho \xi = 0, \tag{5.6.2}$$

$$r_0^{-1} f = \frac{\omega^2}{g_0} - \frac{g_0 k^2}{\omega^2}, \tag{5.6.3}$$

where $g_0 = g(r_0)$ and $k = L/r_0$ is the horizontal wave number of the mode. Under these circumstances the discriminant f is constant. For p modes $f > 0$ and for g modes $f < 0$. The f mode satisfies $f = 0$. It is evident that under these circumstances eqs. (5.6.1)–(5.6.3) are satisfied by

$$\delta p = 0, \qquad \xi = e^{k(r-r_0)}, \tag{5.6.4}$$

$$\omega^2 = gk. \tag{5.6.5}$$

The displacement eigenfunction decays exponentially with depth. It therefore increases exponentially with height, which is an indication that the mode is concentrated in the uppermost layers of the star, particularly when k is large. This suggests that $r = r_0$ should be chosen to be near the surface of the star. When k is large, the plane-parallel model is a very good approximation, since then the mode is confined essentially to a very shallow layer, with characteristic depth k^{-1}.

To complete the solution I first point out that in the plane-parallel approximation the spherical harmonics reduce to a linear superposition of plane harmonic functions of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ for horizontal vectors $\boldsymbol{k}$ such that $|\boldsymbol{k}| = k$. Hence, it follows from eq. (3.6) that the amplitude of the horizontal component η of the displacement $\boldsymbol{\xi}$ is

$$\eta = i\xi. \tag{5.6.6}$$

There is a formal companion to the solution (5.6.4)–(5.6.6) of eqs. (5.6.1)–(5.6.3) with the same frequency:

$$\delta p = e^{-k(r-r_0)}, \qquad \xi = e^{k(r-r_0)}\int (g^{-1}k - c^{-2})\rho^{-1} e^{-2k(r-r_0)}\, dr,$$
$$\eta = i(\xi + g^{-1}\rho^{-1} e^{-k(r-r_0)}). \tag{5.6.7}$$

The eigenfunction grows with depth and can exist only if the mode were forced mechanically from below (cf. appendix V). Therefore, in a static star it must be rejected.

The mode described by eqs. (5.6.4)–(5.6.6) is a surface gravity wave. It is identical to the waves on the surface of deep water, such as an ocean. Indeed, both the nature of the fluid and its stratification are irrelevant to the f mode: the solution depends only on g and k. It is easily verified that the flow is irrotational; pressure gradients exactly cancel buoyancy, preventing the generation of vorticity. Thus the oscillation is independent of the density stratification. Moreover, the flow is also solenoidal; although the fluid is compressible, it is actually not compressed, so the motion does not depend on the equation of state. As far as the surface gravity mode is concerned, a star or a deep lake are indistinguishable. I remark that the second solution, eq. (5.6.7), is neither solenoidal nor (unless H is constant and $k = H^{-1}$) irrotational.

The variation of g and the spherical geometry distort the flow associated with the surface gravity mode so that it is not strictly irrotational, and buoyancy resulting from the density stratification modifies the frequency. To estimate the modification, it is convenient to use the variational principle (5.3.1). Any close approximation to the eigenfunctions is adequate. For example, one may choose $\xi = -\mathrm{i}\eta = \mathrm{e}^{L(r-r_0)/r}$, or $\xi = \mathrm{e}^{k(r-r_0)}$ and $\eta = \mathrm{i}L^{-1}(2+rk)\xi$, so that $\operatorname{div}\boldsymbol{\xi} = 0$. In both cases one obtains

$$\frac{\omega^2}{g_0 k} \simeq 1 - 2L^{-1} - 3\frac{\int (r-r_0)\rho\,\mathrm{e}^{2kr}\,\mathrm{d}r}{r_0\int \rho\,\mathrm{e}^{2kr}\,\mathrm{d}r} =: 1 - \epsilon(k), \tag{5.6.8}$$

where the integrals are over the entire star. In obtaining the expression in this form it was assumed that $g \propto r^{-2}$. This is consistent with choosing r_0 to be near the surface of the star. It was also necessary to perform an integration by parts and discard the integrated term $k^{-1}\rho\,\mathrm{e}^{2kr}$ whilst retaining $\int \rho\,\mathrm{e}^{2kr}\,\mathrm{d}r$. If one assumes that the integral extends well out into the upper atmosphere, and into the corona if there is one, this is likely to be a good approximation. Defining $\bar{r}(L)$ to be the average of r weighted by $\rho\,\mathrm{e}^{2kr}$, eq. (5.6.8) can be rewritten as

$$\omega^2 \simeq g(r_0)Lr_0^{-1}[1 - 3(\bar{r}/r_0 - 1) - 2L^{-1}]. \tag{5.6.9}$$

Equation (5.6.8) shows how ω^2 depends on ρ. In particular, it shows that provided l is large, but not too large ($l < R/2H$), the mode depends mainly on conditions in the vicinity of the photosphere. Beneath the photosphere one can approximate the equilibrium state with a polytrope, as in appendix I. Thus, density increases downwards only as a power of depth, z. This is overcome by the exponential decay of the displacement eigenfunction, and the weight function $\rho\,\mathrm{e}^{-2kz}$ in the integrals rapidly drops to zero. Above the photosphere the atmosphere is approximately isothermal, and $\rho \propto \mathrm{e}^{-z/H}$. Thus, provided the density-scale height, H, is less than $(2k)^{-1}$, the integrand again decays away from the photosphere (this determines the upper limit to the degree l alluded to above). The mode is thus confined near the photosphere, which is therefore about the best level at which to choose the value of r_0. Then $r - r_0$ is least where the energy density, $\rho\,\mathrm{e}^{2k(r-r_0)}$, of the mode is greatest, and the right-hand side of eq. (5.6.8) is only weakly dependent on L. Indeed, it is easy to show that in that case $\bar{r}/r_0 - 1 = \mathrm{O}(L^{-1})$.

Moreover, the error in the expression in square brackets in eq. (5.6.9), which can be estimated from the next term in the expansion of eq. (5.3.1), is $\mathrm{O}(L^{-2})$. It is interesting to remark that in that term a dependence on the sound speed appears for the first time.

Equation (5.6.9) can be regarded as expressing the fact that the frequency of the f mode is approximately its plane-parallel value evaluated at the centre of energy, $\bar{r}$, of the mode. Provided $\bar{r}/r_0 - 1$ is small, the equation can be rewritten as

$$\omega^2 = [1 - 2L^{-1} + \mathrm{O}(L^{-2})]g(\bar{r})L/\bar{r}. \tag{5.6.10}$$

There is interesting behaviour at high L that is worth mentioning. As L increases, the more rapid exponential decay of the energy density of the oscillation with depth causes $\bar{r}$ to rise. Consequently ω^2 falls, and appears to approach asymptotically a value of gk characteristic of the photosphere. But as k approaches $k_c = (2H)^{-1}$, there is no longer a rapid decay of the energy density with height in the atmosphere. Once k exceeds k_c the centre of energy moves upwards as k increases. In the case of the Sun, the movement is halted by the low coronal density. Thus ω^2/L decreases further, essentially to the value of g/r at the chromosphere–corona transition. This behaviour is illustrated in fig. 4. If ever such modes could be observed in the Sun, their frequencies might in principle provide a direct mechanical means of determining the height h (not to be confused with the scale height of g/r^2 defined by eq. (5.4.3)) of the base of the corona above the photosphere. In practice, however, they are likely to be masked by the irregularities in the chromosphere. Moreover, if the modes are excited by mechanical interactions in the convection zone, they might perhaps also be masked by modes with a somewhat higher frequency satisfying eq. (5.6.7) (see appendix V).

Equation (5.6.8) is a relation between the deviation $\epsilon(k)$ of ω^2/g_0k from unity and the Laplace transform of the density,

$$\hat{\rho}(p) := \int_0^\infty \rho(z)\,\mathrm{e}^{-pz}\,\mathrm{d}z, \tag{5.6.11}$$

and one might therefore hope to use it to measure the functional form of ρ in the Sun. (For the rest of this section p is the independent variable of the Laplace transform, and not pressure.) Now $z = r_0 + h - r$ is the depth beneath the base of the corona; I am taking $r = r_0$ to be the photosphere and I am assuming L to be small enough for the contributions to the integrals in eq. (5.6.8) from the corona to be negligible. Noting that the derivative of $\hat{\rho}$ with respect to p is an integral over z weighted in a manner appropriate for evaluating the numerator in the last term in eq. (5.6.8), the equation can be recast as

$$\frac{\mathrm{d}\ln\hat{\rho}}{\mathrm{d}p} = \frac{r_0}{3}\left(1 - \frac{3h}{r_0} - \frac{4}{r_0 p} - \frac{2\omega^2}{g_0 p}\right). \tag{5.6.12}$$

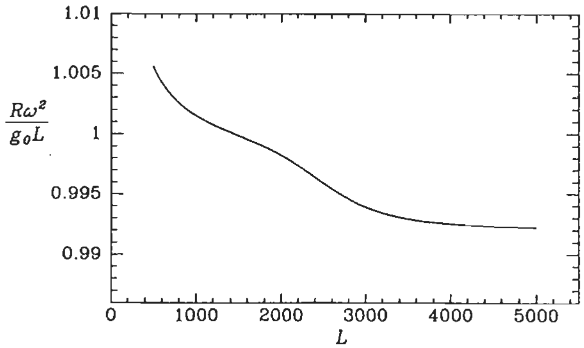


Fig. 4. Frequencies of high-degree f modes given by eqs. (5.6.4)–(5.6.6), drawn schematically for an idealized model of the solar envelope and atmosphere (a plane-parallel isothermal atmosphere over a polytropic interior, meeting at $r = R$, as in appendix I, supporting a high-temperature, isothermal corona) and showing the transition from values of l for which the energy density is concentrated near the photosphere, to those where it is concentrated immediately beneath the corona.

It is now convenient to express ω^2 in terms of the plane-parallel value and its deviation $\epsilon(k)$ defined by the second part of eq. (5.6.8). Then eq. (5.6.12) can be integrated to

$$\hat{\rho}(p) = \hat{\rho}_0 p^{-4/3} \exp\left[-hp + \tfrac{2}{3} r_0 \int^{1/2p} \epsilon(k)\, \mathrm{d}k\right], \tag{5.6.13}$$

providing the Laplace transform $\hat{\rho}$ in terms of an observable function. The constant $\hat{\rho}_0$ is undetermined, as it must be since the right-hand side of eq. (5.6.8) is unaltered by rescaling ρ with a constant factor. One might thus hope to determine ρ up to a constant factor by inverting the Laplace transform.

5.7. *Modes of high degree*

I define these modes simply according to the criterion $l \gg 1$. In all practical cases, provided $n \ll l$, they are confined within a spherical shell whose thickness is much less than the radius of the star.

5.7.1. *Subphotospheric modes*

The discussion in section 5.5 showed how high-degree p modes are confined to the outer layers of a star. I will discuss this confinement in terms of acoustic refraction in section 8. Meanwhile I simply accept the result, and in order to estimate the eigenfrequencies I model only the outer layers of the stellar envelope. I use the

plane-parallel polytrope with index μ described in appendix I. This problem is discussed by Lamb (1932) in a different context. Equation (5.4.7) still describes the motion, and eqs. (5.4.8), (5.4.9) and (5.4.5), determining the vertical component of the wave number K, simplify to

$$K^2 + k^2 = \left[\frac{\mu+1}{\gamma g}\omega^2 + \left(\mu - \frac{\mu+1}{\gamma}\right)\frac{gk^2}{\omega^2}\right] z^{-1} - \frac{1}{4}\mu(\mu+2)\, z^{-2}, \tag{5.7.1}$$

where z is the depth coordinate. The substitution $\Psi = \rho^{1/2} c^2 \mathrm{e}^{-kz} \chi \propto z^{1+\mu/2} \times \mathrm{e}^{-kz} \chi$, $z = (2k)^{-1}\zeta$ then transforms eq. (5.4.7) into the confluent hypergeometric equation

$$\zeta \frac{\mathrm{d}^2\chi}{\mathrm{d}\zeta^2} + (\mu + 2 - \zeta)\frac{\mathrm{d}\chi}{\mathrm{d}\zeta} + a\chi = 0, \tag{5.7.2}$$

where

$$2a = \frac{\mu+1}{\gamma}\sigma^2 - (\mu+2) + \left(\mu - \frac{\mu+1}{\gamma}\right)\sigma^{-2}, \tag{5.7.3}$$

with

$$\sigma^2 = \omega^2/gk. \tag{5.7.4}$$

One must choose the solution such that Ψ does not diverge as $\zeta \to \infty$; then Ψ actually decays (exponentially) at great depth, as the preceding discussion requires. In the notation of Abramowitz and Stegun (1964) it is given by

$$\chi = \chi_0 U(-a,\ \mu+2,\ 2kz), \tag{5.7.5}$$

where χ_0 is an arbitrary constant. Provided the quantity a is not very large and positive (i.e. σ^2 is not very large, or σ^2 is not very small with β, defined by eq. (5.7.8), positive), the upper turning point, above which the mode is evanescent, is not very near the surface, and the mode is therefore only very weakly influenced by conditions in the atmosphere. Therefore I can safely approximate the envelope by the complete polytrope, extending upwards right to $z = 0$, which is the regular singular point of the confluent hypergeometric equation. The condition that U does not diverge at $z = 0$ gives that a should be a non-negative integer $n - 1$, in which case $U = (-1)^n (n-1)!\, L_{n-1}^{(\mu+2)}(\zeta)$, where L is the (generalized) Laguerre polynomial. Hence,

$$\frac{\mu+1}{\gamma}\sigma^2 - \mu + \left(\mu - \frac{\mu+1}{\gamma}\right)\sigma^{-2} = 2n, \quad n = 1, 2, 3, \ldots. \tag{5.7.6}$$

As with the isothermal atmosphere, this is a quadratic equation in ω^2, with p-mode and g-mode roots. For p modes σ^4 is quite large compared with unity, and the third term on the right-hand side of eq. (5.7.3) is small. Then

$$\frac{\omega^2}{gk} \simeq \frac{2\gamma}{\mu+1}(n+\mu/2) - \frac{\beta}{2(n+\mu/2)}, \tag{5.7.7}$$

where

$$\beta = \mu - \frac{\mu+1}{\gamma} = \frac{N^2 z}{g}. \tag{5.7.8}$$

A plot of the frequency, $\omega(k)$, is thus essentially a sequence of parabolae with apices at the origin. Actually, in a star like the Sun, with an outer convection zone, the stratification is almost adiabatic, so β is small, and the buoyancy correction term is extremely small indeed. Since the envelope is really neither plane parallel nor polytropic, eq. (5.7.7) is not strictly satisfied. Nevertheless, the approximately parabolic appearance of the dispersion relation persists (fig. 5).

If the outer layers of the star were stably stratified ($N^2 > 0$), the envelope could also support g modes, with frequencies given by the smaller root of eq. (5.7.6):

$$\frac{\omega^2}{gk} \simeq \frac{\beta}{2(n+\mu/2)}\left[1 + \frac{(\mu+1)\beta}{4\gamma(n+\mu/2)^2}\right]. \tag{5.7.9}$$

Equations (5.7.7) and (5.7.9) approximate the p-mode and g-mode eigenfrequencies of order n.

When n is large, the upper turning point for p modes approaches the surface, and it is perhaps wise to refine the equilibrium model by replacing the very outer layers of the polytrope by an isothermal atmosphere (see appendix I). Then the boundary condition (5.2.8) (or, equivalently, eq. (A5.7)) must be applied to the solution (5.7.5). The resulting eigenfrequency equation is complicated (appendix VI), but, provided $Hk \ll 1$ in the atmosphere, the discontinuity in the stratification lies in the evanescent region and the correction to eq. (5.7.6) is small; when n is large the eigenvalue equation can be simplified to

$$\sigma^2 = \frac{\omega^2}{gk} \simeq s_n^2 - \frac{2\gamma}{(\mu+1)^2\Gamma(\mu+2)\Gamma(\mu+3)}\left[\frac{1}{2}(\mu+1)\frac{\omega}{\omega_{c0}}\right]^{2(\mu+2)}, \tag{5.7.10}$$

where ω_{c0} is the acoustical cutoff frequency in the atmosphere and $s_n \simeq [2n\gamma/(\mu+1)]^{1/2}$ solves eq. (5.7.6) for σ and specifies the frequency of the p mode of order n in the complete polytrope. The modification of that value due to the atmosphere

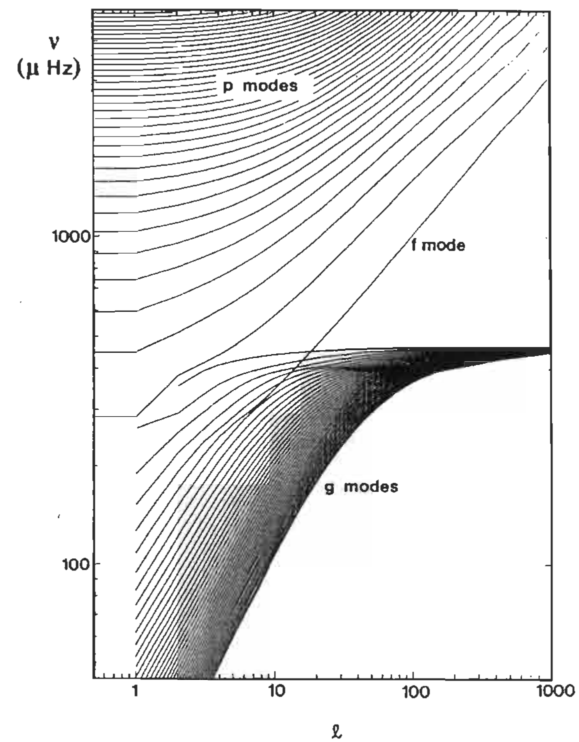


Fig. 5. Cyclic frequencies ν of modes of a solar model, plotted against degree l. Points associated with modes of the same order n are joined by straight lines. When l is large, the modes separate into two distinct groups. The modes in the higher-frequency group are the p and f modes. When $l \gg n$, their $\nu - l$ relation is roughly parabolic, as is predicted by the analysis of the oscillations of a plane-parallel polytrope. For low l, the p and f modes are not so severely concentrated in the outer layers of the star, and the p-mode frequencies tend to finite values as $l \to 0$, which for large n are given by an asymptotic formula of the form (4.8.45) (cf. eq. (5.8.31)). The lower-frequency group contains the g modes. (For clarity, g modes with $|n| > 40$ are not plotted.) Because the solar convection zone is essentially adiabatically stratified almost throughout, $\beta \simeq 0$ and the analysis of section 5.7.1 does not apply. Instead, the modes are confined beneath the convection zone: for $l \gg n$ the frequencies tend to a finite limit, given by eq. (5.7.17), and for $n \gg l$ they are given by eq. (5.8.34), varying almost linearly with l and inversely proportional to n (from Christensen-Dalsgaard (1986)).

is thus a strong function of frequency. Observations indicate that in the surface layers of the Sun, e.g., the effective polytropic index is about 3 (see section 6), and hence at fixed k, $(\sigma^2 - s_n^2)/s_n^2 \propto (\omega/\omega_{c0})^{2(m+1)} \simeq (\omega/\omega_{c0})^8$.

For an adiabatically stratified plane-parallel polytrope, whose index is $\mu = (\gamma-1)^{-1}$, eq. (5.7.6) simplifies to $\omega^2 = 2(n+\mu/2)\mu^{-1}gk$. It has been remarked that this relation includes the f mode, since it implies $\omega^2 = gk$ when $n = 0$. It has been concluded that in view of this result the f mode must be the fundamental p mode. It is obvious both from the analysis in this section and from the discussion in section 5.6 that that conclusion is both mathematical and physical nonsense. Perhaps the most immediate reason is that $\omega^2 = gk$ when $n = 0$ is a formal solution of the quadratic equation (5.7.6), with the definition (5.7.4), only if the envelope is adiabatically stratified; otherwise the roots are different. But more important is that condition (5.7.6) holds only for $n \geqslant 1$; if n were zero, $U(-a, \mu + 2, 2kz)$ would diverge as $z \to 0$, which is not permitted. Of course one might then argue that, in that case, to prevent the divergence of χ one must set $\chi_0 = 0$, implying $\chi = 0$ everywhere, which is indeed a property of the f mode. However, in its raw form that reasoning is obviously bogus because it could be applied for any value of ω. One cannot deduce anything about the eigenvalues from the trivial solution of eq. (5.7.2) alone; it is necessary to find a nontrivial solution of the full pulsation equations (5.1.9) and (5.1.10) in the Cowling approximation that is consistent with that trivial solution for χ. As was demonstrated in section 5.6, such a solution exists only when $\omega^2 = gk$. For the mathematically minded, the argument is thus complete. But I wish to add the physically compelling remark that when $\chi = 0$ there is no compression, nor rarefaction: the very essence of acoustics is therefore totally absent, so the high-degree f mode cannot possibly be an acoustic oscillation.

5.7.2. Atmospheric modes

In the other limit, $Hk \gg 1$, the situation for p modes is quite different. Now the penetration of the mode into the polytropic interior is negligible: the mode resides mainly in the atmosphere and is essentially a horizontally propagating acoustic wave. Provided the star has a corona, or at least a temperature minimum, the greater sound speed both high in the atmosphere and deep in the subphotospheric layers renders the atmosphere into a wave guide by refracting the waves back into the cooler regions. The dispersion relation is then given by

$$\frac{\omega^2}{c^2} \simeq k^2 + \frac{(n-\epsilon)^2\pi^2}{h^2}, \qquad (5.7.11)$$

where h is the height of the corona above the photosphere and c is the mean sound speed in the atmosphere. The phase factor ϵ is determined by the degree of penetration of the mode into the corona and into the (polytropic) interior. It is evidently positive, since the penetration increases the effective vertical extent of the wave guide, and decreases with increasing k. These modes are the closest approximation one can find to Lamb waves (see appendix VI) in a real stellar atmosphere.

A g mode can also be channelled in the atmosphere between the corona and the

subphotospheric layers where N^2 is small. Its frequency is given by

$$\omega^2 \simeq \frac{h^2k^2N^2}{h^2k^2 + (n-\epsilon)^2\pi^2}. \qquad (5.7.12)$$

The phase factor ϵ arises here in just the same way as it did for the p modes. It is not given by the same formula, of course, though it does decrease with k.

5.7.3. Interior g modes

For $l \to \infty$, the critical frequency ω_-, given by eq. (5.5.2), tends to $\mathcal{N}$, and g modes are trapped near $r = r_m$, the locations of the local maxima $\mathcal{N}_m$ of $\mathcal{N}$. This has been discussed by Christensen-Dalsgaard (1980). After expanding $\mathcal{N}$ about $\mathcal{N}_m$, eq. (5.4.7) becomes

$$\Psi'' - \frac{L^2}{r_m^2}\left[\frac{b+2}{b+3} - \frac{b}{\beta} + (b+3)\left(\frac{r - r_m - r_0}{r_m}\right)^2\right]\Psi \simeq 0, \qquad (5.7.13)$$

where here

$$\beta = -\frac{r_m^2}{\mathcal{N}_m}\frac{d^2\mathcal{N}}{dr^2}\bigg|_{r=r_m}, \qquad b = \beta\frac{\mathcal{N}_m^2}{\omega^2} \qquad (5.7.14)$$

and

$$r_0 = \frac{r_m}{b+3}. \qquad (5.7.15)$$

By inspection of the coefficient of Ψ in eq. (5.7.13) it is evident that for low-order modes, $\omega^2 \sim (3+b)\mathcal{N}_m^2/(2+b)$ and that the eigenfunction Ψ is concentrated about $r = r_m + r_0$, somewhat above the position $r = r_m$ of the maximum of $\mathcal{N}$. If the peak in $\mathcal{N}_m^2$ is sharp, then $b \gg 1$ and $r_0 \ll r_m$; the geometrical term arising from the variation in L^2/r^2 being small and $\omega^2 \sim \mathcal{N}_m^2$ as $L \to \infty$. The solutions of eq. (5.7.13) are parabolic cylindrical functions, which are regular as $L^2(r - r_m - r_0)^2/r_m^2 \to \infty$, provided

$$\frac{L}{2\sqrt{b+3}}\left(\frac{b}{\beta} - \frac{b+2}{b+3}\right) \sim 2n' - 1, \quad n' = 1, 2, 3, \ldots. \qquad (5.7.16)$$

Evidently $-n'$ is the order n of the mode. Thus

$$\omega^{-2} \sim \mathcal{N}_m^{-2}\left[\frac{b+2}{b+3} + \frac{(2|n|-1)\sqrt{b+3}}{L}\right], \qquad (5.7.17)$$

where now $b \simeq \frac{1}{2}(\beta - 3 + \sqrt{\beta^2 + 2\beta + 9})$. Moreover, the modes are concentrated within $\pm r_m\delta$ around $r = r_m + r_0$, where

$$\delta \sim \left(\frac{2|n| - 1}{L\sqrt{b+3}}\right)^{1/2}. \tag{5.7.18}$$

Of course, since the modes are confined to a very thin layer inside the star, one cannot expect to observe them at the surface.

5.8. *Modes of high order*

When n is large, the characteristic scale of variation of Ψ is generally much less than the scale heights of the equilibrium state, and the JWKB approximation can be applied. Equation (5.4.7) subject to the condition that $\Psi \to 0$ as $r \to 0$ (which follows from the transformation (5.4.6) and the regularity condition (5.2.2)) and the surface boundary condition (5.2.8) or, more appropriately, an analogous condition of the form $\Psi' + \bar{\kappa}\Psi = 0$ at $r = R$, is formally the same problem as eq. (4.8.5) subject to the condition $\Xi \to 0$ as $r \to 0$ (which follows from the transformation (4.8.4) and the regularity condition (4.2.4)) and the surface boundary condition (4.8.36). The principal difference is between the definitions (5.4.8) and (4.8.6) of K^2. However, since $K^2 = \mathrm{O}(r^{-2})$ as $r \to 0$ in both cases, the two problems have essentially the same mathematical structure, save for the difference in the dependence of the quantities K^2 and $\bar{\kappa}$ on ω. Therefore the solutions (4.8.33) and (4.8.37)–(4.8.39) hold for Ψ, and the eigenvalues are given by eq. (4.8.40). Explicitly,

$$\int_{r_1}^{r_2} \left[\frac{\omega^2 - \omega_c^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{\mathcal{N}^2}{\omega^2}\right)\right]^{1/2} \mathrm{d}r \sim (n - \tfrac{1}{2})\pi + \tan^{-1}\epsilon, \quad n = 1, 2, 3, \dots, \tag{5.8.1}$$

where ω_c^2 and $\mathcal{N}^2$ are given by eqs. (5.4.9) and (5.4.3)–(5.4.5) and ϵ is defined similar to eq. (4.3.37). Strictly speaking, this equation is valid only for large n. For the case of p modes the term $\omega^{-2}\mathcal{N}^2$ can be ignored to a first approximation, and eq. (5.8.1) reduces to

$$\frac{\pi(n + \tilde{\alpha})}{L} \simeq \frac{\omega}{L}\int_{r_1}^{r_2}\left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2 r^2}\right)^{1/2}\frac{\mathrm{d}r}{c}. \tag{5.8.2}$$

For high-order g modes

$$\frac{\pi(n + \tilde{\alpha})}{L} \simeq \frac{1}{\omega}\int_{r_1}^{r_2}\left(1 - \frac{\omega^2}{\mathcal{N}^2} - \frac{\omega_c^2}{\mathcal{N}^2}\frac{\omega^2 r^2}{L^2c^2}\right)^{1/2}\frac{\mathcal{N}}{r}\,\mathrm{d}r, \tag{5.8.3}$$

where

$$\tilde{\alpha} = \pi^{-1}\tan^{-1}\epsilon - \tfrac{1}{2}. \tag{5.8.4}$$

In both cases the turning points r_1 and r_2 are to be interpreted as the zeros of the respective integrands.

I wish also to record the eigenfunctions in the region of propagation ($K^2 > 0$). The Lagrangian pressure perturbation is obtained from Ψ using the transformation (5.4.6):

$$\delta p \sim \Psi_0 u K^{-1/2}\sin\left(\int_{r_1}^{r} K\,\mathrm{d}r + \frac{\pi}{4}\right), \tag{5.8.5}$$

where

$$u = (r^{-3}g\rho f)^{1/2} \tag{5.8.6}$$

and Ψ_0 is a constant. The amplitudes (ξ, η) of the displacement eigenfunction can then be obtained from the governing differential equations. The radial component can be obtained directly from eq. (5.1.10) with Φ' neglected. For p modes it is then most convenient to obtain η from the horizontal component of the momentum equation (3.6). That is not a convenient route for the g modes, because when n is large, $p' = \delta p + g\rho\xi$ is small and it is troublesome to evaluate a small difference between relatively large quantities. It is more expedient to work from the continuity equation (3.10) and the equation of state (3.11); this route is not suitable for p modes, because for them one encounters severe cancellation. The expressions for the amplitudes for p modes are (cf. Shibahashi (1979)):

$$\xi \sim \frac{\Psi_0 K^{1/2}}{r^2 u}\cos\left(\int_{r_1}^{r} K\,\mathrm{d}r + \frac{\pi}{4}\right), \tag{5.8.7}$$

$$\eta \sim \frac{\Psi_0 L u}{\omega^2 r\rho K^{1/2}}\sin\left(\int_{r_1}^{r} K\,\mathrm{d}r + \frac{\pi}{4}\right). \tag{5.8.8}$$

For g modes they are

$$\xi \sim \frac{\Psi_0 L^2 g}{\omega^2 r^4 u K^{1/2}}\sin\left(\int_{r_1}^{r} K\,\mathrm{d}r + \frac{\pi}{4}\right), \tag{5.8.9}$$

$$\eta \sim \frac{\Psi_0 L g K^{1/2}}{\omega^2 r^3 u}\cos\left(\int_{r_1}^{r} K\,\mathrm{d}r + \frac{\pi}{4}\right). \tag{5.8.10}$$

Notice from the approximation (5.8.2) that in the case of p modes

$$\frac{r_1}{c(r_1)} \simeq \left(1 + \frac{\omega_c^2(r_1)}{2\omega^2}\right)\frac{L}{\omega}. \tag{5.8.11}$$

This shows explicitly how the radius of the lower turning point increases with l at fixed frequency.

From the approximate solutions (5.8.7)–(5.8.10) one can evaluate the asymptotic displacement amplitude ratios. For p modes

$$\left|\frac{\xi}{\eta}\right| \sim \frac{\omega^2 r^2 K}{Lgf} \sim \frac{rK}{L}. \tag{5.8.12}$$

Except near the turning points, where K vanishes, K is dominated by the first term of eq. (5.4.8): $\omega/c \gg L/r$ when $n \gg 1$, and therefore $|\xi/\eta| \gg 1$. The motion of high-frequency p modes is almost vertical everywhere, except in the vicinity of the turning points, where it is nearly horizontal. The reason for this will become clear in the light of the discussion of acoustic ray paths in section 8. For g modes

$$\left|\frac{\xi}{\eta}\right| \sim \frac{L}{rK}. \tag{5.8.13}$$

Near the turning points the motion is nearly vertical. But well in the region of propagation

$$rK \sim L\mathcal{N}/\omega \gg 1, \tag{5.8.14}$$

and consequently $|\xi/\eta| \sim \omega/\mathcal{N}$. The value of the ratio depends on the relative values of n and l. More precisely, it depends on the local ratio of the horizontal wave number, L/r, to the vertical wave number, K, as eq. (5.8.13) makes quite clear. The formula follows from the fact that δp is small, and that therefore the magnitude of div $\boldsymbol{\xi}$ is much smaller than its constituent terms. Since the variation of $\mathcal{N}$ (or, more precisely, ω_-) is typically quite smooth (see fig. 2), $L/rK \sim l/n$: the eddy motion of g modes is mainly vertical when $l/n \gg 1$ and mainly horizontal when $l/n \ll 1$.

The reason I have laboured on the discussion of the amplitude ratio for g modes is that there seems to be a myth pervading the literature that the motion of g modes is always nearly horizontal, which is clearly not the case. It appears from Cox (1980) that the misunderstanding might have arisen from not reading Cowling's (1941) first discussion on the subject with sufficient care. Cowling, quite correctly, showed that $|\xi/\eta| \ll 1$ when $l/n \ll 1$, but the condition on l/n has often been overlooked since.

The amplitude ratio of p-mode displacements depends also on the local ratio of vertical and horizontal wave numbers, but in the opposite sense. The reason why the conclusion is not merely the inverse of that at which I arrived for g modes is that, unlike $\mathcal{N}$, the Lamb frequency, S_l, varies steeply throughout the star, and very quickly becomes small compared with $\omega/c \simeq K$ above the lower turning point. Therefore one cannot equate L/rK with l/n: indeed, away from the lower turning point the value of l is hardly relevant and all acoustic modes look like radial modes. These remarks are correct, of course, only if l is not so large as to dominate K throughout the region of propagation, producing the atmospheric pseudo-Lamb wave discussed in section 5.7.2.

It is also worth noting that the phase relations exhibited by the asymptotic solutions are just as one would expect from the basic physics of acoustic waves and internal gravity waves. In acoustics, the pressure fluctuation, δp, is generated directly by the compression and rarefaction of the dominant motion, which, except near the turning points, is nearly vertical. Therefore δp is in phase with $\partial\xi/\partial r$, and $\frac{1}{2}\pi$ out of phase with ξ. The horizontal displacement must be $\frac{1}{2}\pi$ out of phase with ξ and in phase with δp to maintain continuity. In the case of g modes, the motion is so slow that the pressure has plenty of time to readjust (by acoustic communication); the Eulerian pressure fluctuation, p', is small and consequently the (Lagrangian) pressure in a moving fluid element is determined simply by the pressure of the environment into which it has been displaced. Since the pressure in the equilibrium state varies only in the vertical direction, δp is in phase with ξ. Once again, η must be $\frac{1}{2}\pi$ out of phase to maintain continuity.

It is instructive to look in a little more detail at the asymptotic relations for large and small l. I will restrict the attention to p modes. In the case of small l the lower turning point $r = r_1$ is close to the singularity at $r = 0$. Evaluating the exponentially decaying branch of the appropriate Airy-function representation (cf. section 4.8.4) yields $\delta p \propto r^{L-1/2}$ and $\xi \propto r^{L-3/2}$. These forms are close to, but not precisely equal to, the correct values obtained by analyzing the behaviour of the solution in the vicinity of $r = 0$. It is interesting to note, however, that if $L = \sqrt{l(l+1)}$ is replaced by $l+\frac{1}{2}$, then the Airy-function representation yields the precise behaviour (5.2.2)–(5.2.3) when $l \neq 0$. It makes very little difference to the solution when l is large. As I am about to demonstrate, the replacement of L by $l+\frac{1}{2}$ also improves the estimates of the eigenvalues; it is a general feature of the JWKB representation of the operator ∇^2, which occurs in the acoustic wave equation, and appears to have been noticed originally in connection with the asymptotic solution of Schrödinger's equation for a Coulomb potential (e.g. Kemble (1937)).

One can investigate the behaviour of the solution near $r = 0$ by ignoring the small term ω_c^2/ω^2 in the p-mode approximation to K used in eq. (5.8.2). That term is important only near the surface (ω_c^2 is bounded as $r \to 0$), and is the sole term in the eigenvalue equation that depends on the density stratification. One can simplify the problem still further by letting the sound speed be constant and replacing condition (5.2.8) by a simpler condition such as $\delta p = 0$ at $r = R$. The simplification of the analysis is substantial, but since $dc^2/dr \to 0$ as $r \to 0$ in any star, it does not significantly alter the mathematical structure of the approximation. The problem is now reduced to that of finding the eigenfrequencies of an isothermal sphere in the absence of gravity, with a constant pressure applied to the bounding surface, whose unperturbed position is $r = R$. That problem is solved in appendix VII. In the limit $n/l \to \infty$, the eigenfrequencies are given approximately by

$$\omega \sim cR^{-1}(n + \tfrac{1}{2}l), \tag{5.8.15}$$

whereas the expansion of eq. (5.8.2) in that limit yields

$$\omega \sim cR^{-1}(n + \tfrac{1}{2}L - \tfrac{1}{4}). \tag{5.8.16}$$

These two formulae become identical when L is replaced by $l + \frac{1}{2}$.

When $l/n \gg 1$, p modes are confined to a relatively shallow subphotospheric layer, and, as usual, one can approximate the equilibrium state of the envelope by the complete plane-parallel polytrope with index μ described in appendix I. In this case $\epsilon = 0$, and eq. (5.8.1) reduces to

$$\begin{aligned}(n - \tfrac{1}{2})\pi \\ &= \int_{z_1}^{z_2} \left[\frac{(\mu+1)\omega^2}{\gamma g z} + \left(\mu - \frac{\mu+1}{\gamma}\right)\frac{gk^2}{\omega^2 z} - \frac{(\alpha+\frac{1}{2})^2}{z^2} - k^2\right]^{1/2} dz \\ &= \frac{1}{2}\left[\frac{(\mu+1)\omega^2}{\gamma g k} + \left(\mu - \frac{\mu+1}{\gamma}\right)\frac{gk}{\omega^2} - 2\alpha - 1\right]\pi \end{aligned} \tag{5.8.17}$$

(cf. eq. (5.7.1)), where z_1 and z_2 are the upper and lower turning points and

$$\alpha = \tfrac{1}{2}[\mu(\mu+2)]^{1/2} - \tfrac{1}{2}. \tag{5.8.18}$$

The eigenvalue equation (5.8.17) can be rewritten as

$$\frac{\mu+1}{\gamma}\sigma^2 - 2\alpha + \left(\mu - \frac{\mu+1}{\gamma}\right)\sigma^{-2} = 2n, \tag{5.8.19}$$

where $\sigma^2 = \omega^2/gk$. This equation would be made identical to the exact result (5.7.6) if α were replaced by $\frac{1}{2}\mu$, a replacement which is similar to that suggested for radial modes at the end of section 4.8.5. It is equivalent to replacing

$\omega_c^2/c^2 = \frac{1}{4}\mu(\mu+2)z^{-2}$ by $\frac{1}{4}(\mu+1)^2 z^{-2} = \frac{1}{4}H_p^{-2}$. As with the expansion near the singular point at $r = 0$, this change is quite minor. Note that as μ increases, the quantity α, defined by eq. (5.8.18), approaches its exact value $\frac{1}{2}\mu$. It is now straightforward to evaluate the depths of the turning points. These are given by

$$z_{1,2} = RL^{-1}\{n + \tfrac{1}{2}\mu \pm [(n + \tfrac{1}{2}\mu)^2 - \tfrac{1}{4}(\mu+1)^2]^{1/2}\}, \tag{5.8.20}$$

in which I have set $\alpha = \frac{1}{2}\mu$. In view of the equivalence of eqs. (5.8.19) and (5.7.6), this result can be used even when n is not large.

It is also interesting to note that for the plane-parallel polytrope the effect of ω_c is equivalent to subtracting $(\alpha + \frac{1}{2})\pi$ from the left-hand side of eq. (5.8.17). This provides the basis of an expansion of eq. (5.8.2) about the polytropic value ω_{cp} of ω_c. One is tempted to write, e.g.,

$$\begin{aligned}\frac{\pi(n+\alpha)}{\omega} &\simeq \int_{r_1}^{R}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{1/2}\frac{dr}{c} \\ &\quad - \frac{1}{2\omega^2}\int_{r_1}^{R}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2}\frac{\omega_c^2 - \omega_{cp}^2}{\omega_{cp}^2}\frac{dr}{c},\end{aligned} \tag{5.8.21}$$

where ω_{cp} is the critical frequency of a polytropic envelope whose structure approaches that of the stellar envelope well beneath the photosphere, and the lower turning point r_1 now satisfies $\omega r_1 = Lc(r_1)$. The second integral on the right-hand side is small. Notice that I have now replaced the upper limit of integration by R, my "surface" of the star, for the same reason as when I had removed the critical frequency, ω_c, from the integrand in eq. (5.8.17) then it would be necessary to replace z_1 by zero. I have never defined my surface precisely, because I wished to retain the flexibility of choosing it to be different in different discussions to suit my purposes. Here I have in mind essentially the position where c^2 would have vanished had it been extrapolated linearly outwards from the polytropic subphotospheric regions. Since $L^2c^2/r^2\omega^2 \ll 1$ in the atmosphere, any change in the definition of R would add a term to α that is proportional to ω. The reason for expanding the integral in this way is that the leading integral is invertible and therefore provides a means of inferring the distribution of the sound speed throughout a star from knowledge of the frequencies of oscillation. I will discuss this inversion in section 6. I will also have more to say about the weighting factor $(1 - L^2c^2/\omega r^2)^{-1/2}$ when I discuss in section 8 the eigenfunctions in terms of interference between locally plane acoustic waves.

It is particularly interesting to expand the eigenvalue equation (5.8.2) for the case of small l. The reason is that only the low-degree modes are likely to be observed in the immediate future in stars other than the Sun. The leading-order terms are of the same form as eq. (5.8.16), even though c is not constant and ω_c^2 is included. I will take the expansion up to the next term. The principle is to note that near the lower turning point, r_1, where L^2c^2/ω^2r^2 is comparable with unity, c^2 varies slowly since r_1 is small and $dc/dr = 0$ at $r = 0$. Also, $\omega_c^2/\omega^2 \ll 1$. Therefore, using the acoustical radius

$$\tau := \int_0^r \frac{dr}{c} \tag{5.8.22}$$

as the independent variable, one can expand the integrand about $(1 - L^2/\omega^2\tau^2)^{1/2}$ in the inner regions of the star:

$$\begin{aligned} I_1 &:= \int_{\tau_1}^{\tau_m} \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2r^2}\right)^{1/2} d\tau \simeq \int_{L/\omega}^{\tau_m} \left(1 - \frac{L^2}{\omega^2\tau^2}\right)^{1/2} d\tau \\ &- \int_{L/\omega}^{\tau_m} \left(1 - \frac{L^2}{\omega^2\tau^2}\right)^{-1/2} \left(\frac{L^2\psi}{\omega^2\tau^3} + \frac{\omega_c^2}{2\omega^2}\right) d\tau, \end{aligned} \tag{5.8.23}$$

where

$$\psi = \tau - r/c = \int_0^\tau \frac{r}{c}\frac{dc}{dr}\, d\tau, \tag{5.8.24}$$

$\tau_1 = \tau(r_1)$ and τ_m is some median value of τ between the centre and the surface of the star. Note that the integrand in the second term of the expansion (5.8.23) diverges at the turning point $\tau = L/\omega$. Nevertheless the integral is much smaller than the leading term, and the expansion does indeed provide a good approximation to the integral I_1. One can convince oneself of this by expanding ψ/τ^3 and ω_c^2 in a Taylor series about the turning point and retaining the first four terms. For example, the expansion of

$$\int_1^x [1 - (1 + \epsilon x)/x^2]^{1/2}\, dx \tag{5.8.25}$$

in the way of eq. (5.8.23) is identical to order ϵ to the expansion of the exact integral. One cannot take the expansion further, however, since subsequent terms can be infinite. For $\tau > \tau_m$, L^2c^2/ω^2r^2 is much less than unity, and it is therefore a suitable expansion parameter:

$$\begin{aligned} I_2 &:= \int_{\tau_m}^{\tau_2} \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2r^2}\right)^{1/2} d\tau \simeq \int_{\tau_m}^{\tau_2} \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2} d\tau \\ &- \frac{L^2}{2\omega^2} \int_{\tau_m}^{\tau_2} \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1/2} \frac{c^2}{r^2}\, d\tau, \end{aligned} \tag{5.8.26}$$

where $\tau_2 = \tau(r_2)$. After substituting $I_1 + I_2$ into eq. (5.8.2) and rearranging the terms, one can obtain after a further reduction of the second term of the expansion (5.8.26), a formula that is independent of the somewhat arbitrary value τ_m of the acoustic radius for the division between the two regimes of expansion:

$$\tau_2\omega \sim (n + \tfrac{1}{2}L + \tilde{\alpha})\pi - (AL^2 - B)\pi\omega_0/\omega, \tag{5.8.27}$$

where

$$A = \frac{1}{2\pi\omega_0}\left(\frac{c(r_2)}{r_2} - \int_{Lc/\omega}^{r_2} \frac{c}{r}\frac{dc}{dr}\frac{dr}{c}\right), \tag{5.8.28}$$

$$B = \frac{\omega^2}{\pi\omega_0}\int_{Lc/\omega}^{r_2}\left(1 - \frac{L^2c^2}{\omega^2r^2}\right)^{-1/2}\left[1 - \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2}\right]\frac{dr}{c}, \tag{5.8.29}$$

and the characteristic frequency

$$\omega_0 = \pi\left(\int_0^R \frac{dr}{c}\right)^{-1} = \frac{\pi}{\tau(R)} \tag{5.8.30}$$

of section 4.8.5 has been reintroduced. The details of the reductions are presented by Gough (1986). If the outer layers of the star are represented by a polytrope with index μ and the substitutions $L = l + \frac{1}{2}$ and $\omega_c^2/c^2 = \frac{1}{4}H_p^{-2}$ are made, the eigenfrequency equation becomes

$$\omega \sim [n + \tfrac{1}{2}(l + \tfrac{1}{2}) + \alpha]\omega_0 - (AL^2 - \delta)\omega_0^2/\omega, \tag{5.8.31}$$

which generalizes and extends eq. (4.8.45), where α is still given by

$$\alpha = \tfrac{1}{2}\mu \tag{5.8.32}$$

and*

$$\delta = -\frac{(\mu + 1)^2}{2\pi^2}. \tag{5.8.33}$$

It is evident that eqs. (5.8.31)–(5.8.33) also approximate the eigenfrequencies when the outer layers of the envelope in the vicinity of $r = r_2$ are not strictly polytropic, and when μ is now interpreted to be a representative value of $(d\ln p/d\ln\rho - 1)^{-1}$. A formal demonstration of that can be constructed by using Olver's method

* Formula (8.19) in Gough (1986) for δ should read $\delta = -2b^2/\pi^2 = -(2\alpha + 1)^2/2\pi^2$.

with the confluent hypergeometric equation, rather than Airy's equation, as the comparison equation for integrating through the upper turning point.

Equation (5.8.31) is similar to an expression obtained previously by Tassoul (1980), the leading term of which had been obtained earlier by Vandakurov (1967b), although Tassoul's expansion differs in several respects from that presented here. Perhaps the most important is that she did not first cast the equations into the standard form (5.4.7) and consequently it was necessary to carry the Liouville–Green expansion further than I have. Her result for p modes is of the form (5.8.31), but it differs in having the lower limit of integration in eq. (5.8.28) for A set to zero (the difference is of higher order than any of the terms retained and is therefore formally insignificant); moreover the formula for δ is different: if the plane-parallel polytrope is substituted into her formula, one obtains $\delta = (2\mu - 1)(2\mu - 3)/8\pi^2$, which when $\mu = 3$, the value characteristic for the surface layers of the Sun, is only about 30% of the value obtained from eq. (5.8.33).

Despite this discrepancy, it is useful to find the regions of the star that contribute the most to the parameters in the asymptotic expression (5.8.31). The characteristic frequency ω_0 is the characteristic acoustical frequency of the entire star; as has already been pointed out in section 4.8, it is of the same order of magnitude as the frequency ω_0 defined by eq. (4.2.6), which appeared as the natural unit in which to measure the frequencies of the radial modes. The explicit integral in eq. (5.8.30) is weighted most strongly near the surface of the star, where c is small. It is more natural to think of acoustic waves in terms of the variable τ, however, with respect to which the contribution to ω_0 is uniformly distributed. The quantity A depends principally on the conditions near the lower turning point, $r_1 = Lc/\omega$, particularly when considered with respect to τ. This must be the case in the light of my earlier remark that it is only near the lower turning point that K is appreciably dependent on L; it is also evident from formula (5.8.28), which is dominated by an average of the gradient of the sound speed weighted with r^{-1} (irrespective of whether r or τ is the independent variable). The quantities α and δ both depend predominantly on the conditions near the surface: δ is part of B and α contains $\tilde{\alpha}$, the remainder of B, and a term again coming from the surface regions, above the (frequency-dependent) upper turning point τ_2, which emerges from expanding $\pi\tau_2^{-1}$ about $\omega_0 = \pi[\tau(R)]^{-1}$. The phase factor $\tilde{\alpha}$ arises from the upper boundary condition (although it contains a (constant) contribution $-\frac{1}{4}$ coming from the Airy-function representation near the lower turning point) and B comes from the influence of ω_c on K, which is substantial only near the upper turning point.

A similar analysis can be carried out for g modes. Since the details of the results depend on the positions of the convection zones and since the buoyancy frequency at the interface between the convection zone and the radiative zone is probably almost discontinuous when viewed with a resolution comparable with the inverse vertical wave numbers of most g modes of interest, the fine details of the formulae

are more diverse. I record here only that when $n/l \gg 1$, eq. (5.8.3) yields for a star like the Sun, with a radiative interior and a deep, adiabatically stratified convective envelope:

$$P \sim (n + \tfrac{1}{2}l - \tfrac{5}{12})P_0/L, \tag{5.8.34}$$

where $P = \omega/2\pi$ is the period of oscillation, whose natural unit is the characteristic period

$$P_0 = 2\pi^2 \left(\int_0^{r_c} r^{-1}\mathcal{N}\,\mathrm{d}r\right)^{-1}, \tag{5.8.35}$$

where r_c is the radius of the base of the convection zone. For further information the reader is referred to the asymptotic analyses by Zahn (1970), Ledoux and Perdang (1980), Tassoul (1980), Ellis (1986), Gabriel (1986) and Provost and Berthomieu (1986).

Finally it behoves me in this section to remark that for low-degree modes the neglect of the perturbation Φ' to the gravitational potential is not entirely negligible. This is particularly so for g modes which, according to the asymptotic representations (5.8.7)–(5.8.10), induce a larger density fluctuation at fixed mode energy than the p modes do. The effect of Φ' has been incorporated as a small perturbation in the Liouville–Green expansion by W.A. Dziembowski and myself (to be published) in a way not unlike that by which Tassoul (1980) coupled her two formulations of the adiabatic oscillation equation in the Cowling approximation. The principal modification for high-order modes is to subtract $\omega_J^2 := 4\pi G\rho$ from ω_c^2 and to add $\omega_J^2 r/gfH$ to $\mathcal{H}^{-1}$ where it appears in the definitions (5.4.9) and (5.4.5) of ω_c^2 and $\mathcal{N}^2$. The extent of the influence of Φ' on radial modes can be judged from a careful comparison of eq. (5.8.2) with $l = 0$ and eq. (4.8.40), which includes perturbations to the gravitational potential.

6. Inversion of asymptotic formulae

The starting point of this discussion is the approximation to eq. (5.8.21) obtained by neglecting the second integral on the right-hand side:

$$\pi\frac{(n+\alpha)}{\omega} \simeq \mathcal{F}(w) := \int_{r_1}^{R}\left(1 - \frac{a^2}{w^2}\right)^{1/2}\frac{\mathrm{d}r}{c}$$
$$= \int_{r=r_1=c/w}^{R} (a^{-2} - w^{-2})^{1/2}\,\mathrm{d}\ln r, \tag{6.1}$$

where

$$w = \omega/L \tag{6.2}$$

is a reduced frequency, formally equal to the frequency of oscillation of a sinusoidal wave with L wavelengths around a circle rotating with respect to the observer with angular velocity ω, and

$$a = \frac{c}{r} \tag{6.3}$$

is the angular velocity of circular motion with linear velocity c at radius r.

Notice that when $HL/R \ll 1$ in the atmosphere, the quantity α depends essentially only on the frequency. It contains a contribution from the boundary condition via ϵ (defined by eq. (4.8.37) in terms of $\bar{\kappa}$ in section 5.8 and K given by eq. (5.4.8)), which is only very weakly dependent on L, a term resulting directly from the removal of ω_c^2 from the integrand, and a term linear in ω (discussed in section 5.8) that is associated with how R is chosen. Explicitly,

$$\alpha \simeq \tilde{\alpha} + \frac{\omega}{\pi}\left\{\int_{r_1}^{r_2}\left[1-\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}\right]\frac{dr}{c} + \int_{r_2}^{R}\frac{dr}{c}\right\}, \tag{6.4}$$

where $\tilde{\alpha}$ is defined by eq. (5.8.4). In deriving this result I have ignored L^2c^2/ω^2r^2 compared with unity, which is a good approximation in the surface layers near and above $r = r_2$, well beneath which the integrand is small and above which the second integral is evaluated. Neglecting ω_c^2 in the first integral deep in the star where L^2c^2/ω^2r^2 is not negligible, is consistent with ignoring $\mathcal{N}^2$, which in the radiative interior is comparable with ω_c^2. The first integral is small when $\omega \ll \omega_c$ at $r = R$, since then $c(r_2)$ is large, and the second integral is always small, since, when $HL/R \ll 1$, the range of values of R to be considered (measured in units of the acoustical radius, τ) is much less than $\tau(R) - \tau(r_1)$, which estimates the integral on the right-hand side of eq. (6.1). Thus one can conclude that α is a function of ω alone, and, except possibly when $\omega \simeq \omega_c(R)$, it varies only weakly with ω. Moreover, $\pi(n+\alpha)/\omega$ is a function of $w = \omega/L$ alone.

This remarkable result was first noticed empirically by Duvall (1982), from analyzing real solar data. His motivation was to make the dispersion relation for solar oscillations resemble the formula for sound waves travelling in an acoustic wave guide with fixed depth d with uniform sound speed c, for which the dispersion relation is

$$\pi\frac{(n+\tilde{\epsilon})}{\omega} = \left[\frac{1}{c^2} - \frac{1}{(\omega/k)^2}\right]^{1/2} d, \tag{6.5}$$

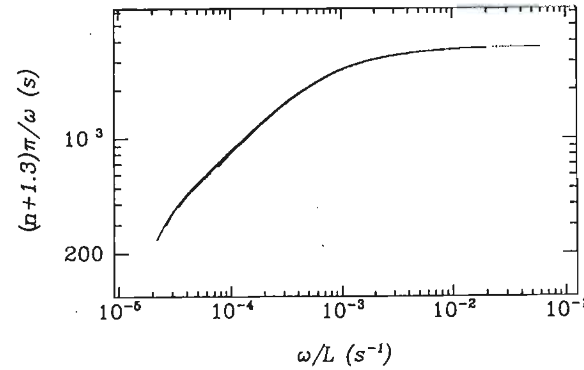


Fig. 6. $(n+\alpha)\pi/\omega$ with $\alpha = 1.3$, plotted against $w = \omega/L$ for p modes, some of whose frequencies are plotted in fig. 5. The units of ω are s^{-1}

where k is the longitudinal wave number and the phase $\tilde{\epsilon}$ depends on the nature of the reflecting walls. This equation has basically the same structure as eq. (6.1), with the right-hand side being a function of d and ω/k alone. When c depends on the crosswise coordinate, the right-hand side of eq. (6.5) is replaced by an appropriate average; and even if the wave is prevented by refraction from reaching a wall, the right-hand side remains a function of ω/k alone, since the crosswise direction of propagation of the wave is reversed at locations where $c = \omega/k$, which is a function only of w. Stratification under gravity does not alter the nature of the result provided buoyancy is small, as the dispersion relation (5.7.7) for a plane-parallel polytrope (with β neglected), which does not rely on asymptotic theory, illustrates. Duvall confirmed the result by plotting $\pi(n+\alpha)/\omega$ for high-degree, five-minute modes against w for different constants α, showing that when $\alpha \simeq 1.5$ the distinct p-mode branches of the dispersion relation illustrated in fig. 5, all fall on a single curve (fig. 6). This is the function $\mathcal{F}(w)$. Since $\alpha \simeq \frac{1}{2}\mu$, Duvall's result also indicates that the effective polytropic index in the subphotospheric layers of the Sun, where five-minute modes are reflected, is about 3. Duvall's technique has been improved by writing $\alpha = \alpha_0 + \alpha_1(\omega)$, where α_0 is a constant, and fitting $\pi(n+\alpha_0)/\omega$ to a sum of a function of ω alone and a function of w alone. That determines not only $\mathcal{F}$, up to an undetermined additive constant, but also α_1, to within an additive linear function of ω. Notice that the integrand in the second term on the right-hand side of eq. (5.8.21) is significant only near $r = R$, where usually $L^2c^2/\omega^2r^2 \ll 1$. Therefore that term is a function of ω alone, and can be considered to have been incorporated into α_1.

The importance of this approximation is that eq. (6.1) can be inverted to yield $a(r)$ as a functional of the observable quantity $\mathcal{F}(w)$. The result immediately provides information on the sound speed $c(r)$ throughout the region spanned by the lower turning points $r_1(w)$. This information is not complete, because the result depends on the undetermined sound speed in the region above the greatest value of r_1, for which data are available.

To carry out the inversion one first differentiates eq. (6.1) with respect to w:

$$\frac{d\mathcal{F}}{dw} = -w^{-3} \int_{a_s}^{w} (a^{-2} - w^{-2})^{-1/2} \frac{d \ln r}{da} \, da, \tag{6.6}$$

where $a_s = a(R)$. One can transform to the independent variable a quite safely provided $a(r)$ is monotonic. This condition is satisfied for stars with unmixed cores, such as late-type main-sequence stars, even though c might decrease towards the centre due to the higher mean molecular mass of the material processed by nuclear reactions. The core of an early-type star is mixed by convection, and there is a near discontinuity in the sound speed. Of course the JWKB approximation cannot be applied across the discontinuity; it is necessary to divide the star into two regions and match across the interface. I do not present those details here.

Equation (6.6) can be cast into Abel's integral equation by the substitutions $a^{-2} = \xi$, $w^{-2} = u$. Then

$$\frac{d\mathcal{F}}{du} = -\tfrac{1}{2} \int_{u}^{\xi_s} (\xi' - u)^{-1/2} \frac{d \ln r}{d\xi'} \, d\xi', \tag{6.7}$$

where $\xi_s = a_s^{-2}$ and I have formally replaced the dummy integration variable ξ by ξ'. This equation is inverted by multiplying both sides of the equation by $(u - \xi)^{-1/2}$ and integrating with respect to u over the range within which $\mathcal{F}$ has been determined from observation. After interchanging the order of the double integration, one obtains

$$\begin{aligned} \int_{\xi}^{u_0} (u-\xi)^{-1/2} \frac{d\mathcal{F}}{du} \, du &= -\tfrac{1}{2} \int_{\xi}^{u_0} \frac{d \ln r}{d\xi'} \, d\xi' \int_{\xi}^{\xi'} [(u-\xi)(\xi'-u)]^{-1/2} \, du \\ &\quad - \tfrac{1}{2} \int_{u_0}^{\xi_s} \frac{d \ln r}{d\xi'} \, d\xi' \int_{\xi}^{u_0} [(u-\xi)(\xi'-u)]^{-1/2} \, du, \end{aligned} \tag{6.8}$$

where $u_0 = w_0^{-2}$ and w_0 is the smallest value of w for which data are available. The first of each of the double integrals is easily calculated with the help of the substitution $u = \xi' \sin^2\theta + \xi \cos^2\theta$. The indefinite integral is simply θ. Hence, after restoring the original variables, and some minor rearrangements, one derives

$$\frac{r}{R} = \exp\left[-\frac{2}{\pi} \int_{w_0}^{a} (w^{-2} - a^{-2})^{-1/2} \frac{d\mathcal{F}}{dw} \, dw - \Lambda(w_0^a) \right], \tag{6.9}$$

where

$$\Lambda = \ln\left(\frac{R}{R_0}\right) - \frac{2}{\pi} \int_{a'=w_0}^{a_s} \sin^{-1}\left(\frac{w_0^{-2} - a^{-2}}{a'^{-2} - a^{-2}}\right)^{1/2} d \ln r', \tag{6.10}$$

$R_0 = r(w_0)$ and $a' = a(r')$. Of course, eq. (6.9) must be evaluated numerically, so it is a trivial matter to invert $r(a)$ to determine, with eq. (6.3), $c(r)$.

Except near $r = R_0$, Λ is approximately constant, and its sole effect is to multiply r/R by a constant scale factor. Provided high-degree data are available, this factor is close to unity, and is then approximately $1 - \Lambda$. For example, if the sound speed in the outer layers of the star above $r = R_0$ is represented by the approximation $c^2 \propto z^{1.2}$, which is suggested from an extrapolation of solar observations, then very roughly $\Lambda \simeq 0.28(n_0 + \alpha)/L_0$, where n_0 and L_0 correspond to the least-deeply penetrating modes (appendix VIII).

Refinements of the procedure are imaginable. For example, if $\mathcal{N}^2/\omega^2$ is treated as a small term in the eigenvalue equation (5.8.1), rather than being neglected, the expansion (5.8.21) becomes, in the variables of eq. (6.1):

$$\pi \frac{(n + \alpha_0)}{\omega} = \mathcal{F}(w) - \frac{\pi \alpha_1(\omega)}{\omega} + \frac{\mathcal{G}(w)}{\omega^2}, \tag{6.11}$$

where

$$\mathcal{G}(w) := \frac{1}{2w^2} \int (a^{-2} - w^{-2})^{-1/2} \mathcal{N}^2 \, d \ln r \tag{6.12}$$

reflects the contamination of the acoustic dynamics by buoyancy and $\alpha_1(\omega)$ is considered to contain the frequency-dependent contribution from the critical acoustic frequency ω_c, represented by the second integral in eq. (5.8.21). Once again the right-hand side of eq. (6.11) has a very special functional form, and therefore in principle it should be possible to determine the functions $\mathcal{F}$, α and $\mathcal{G}$ from the data. The fitting procedure has not yet been carried out with resounding success with real solar data. One might think that if it were possible to so determine $\mathcal{G}(w)$ from observations, then eq. (6.12) could be inverted yielding $\mathcal{N}^2(r)$, since the equation is transformed into eq. (6.6) by the substitution $w^3 \, dF/dw \to -\mathcal{G}(w)$, $d \ln r/da \to \frac{1}{2}\mathcal{N}^2 \, d \ln r/da$, and hence.

$$\mathcal{N}^2 = -\frac{4}{\pi} \frac{d}{d \ln r} \int_{w_0}^{a} w^{-1} (w^{-2} - a^{-2})^{-1/2} \mathcal{G}(w) \, dw + M, \tag{6.13}$$

where M is a quantity analogous to Λ, which takes into account the contribution to $\mathcal{G}$ from regions in the star above $r = R_0$. However, the dominant contribution to eq. (6.11) from the neglected perturbation to the gravitational potential is also of the form ω^{-2} multiplied by a function of w alone, and therefore cannot, at this order of approximation, be separated from the contribution from the buoyancy frequency.

Finally, I remark that if the last term in parentheses in the integrand of eq. (5.8.3) for g-mode frequencies is negligible, then each side is a function of ω alone:

$$\pi\frac{(n+\tilde{\alpha})}{L} \simeq \mathcal{J}(\omega) := \frac{1}{\omega}\int_{r_1}^{r_2}(\mathcal{N}^2-\omega^2)^{1/2}\,\mathrm{d}\ln r. \tag{6.14}$$

This equation can also be cast into Abel form, by a procedure similar to that used for p modes. Now, of course, the natural independent variable $\mathcal{N}^2$ is not monotonic with r; it is necessary to split the range of r into regions within which $\mathcal{N}^2$ is monotonic, and invert each contribution to $\mathcal{J}$ separately. If it is assumed that $\mathcal{N}^2$ has a single maximum $\mathcal{N}_\mathrm{m}^2$ at $r = r_\mathrm{m}$, as in fig. 2, then there are just two regions, and the data $\mathcal{J}(\omega)$ provide information on the distance between the two turning points:

$$r_2(\mathcal{N}) - r_1(\mathcal{N}) = -\frac{2}{\pi}\int_{\mathcal{N}}^{\mathcal{N}_\mathrm{m}}(\omega^2-\mathcal{N}^2)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\mathcal{J})\,\mathrm{d}\omega. \tag{6.15}$$

This information is useful, but alone it is insufficient to determine $\mathcal{N}(r)$ completely.

7. Perturbation theory

In this section I discuss two kinds of perturbation: (a) that arising from small spherically symmetric changes in the equilibrium structure of the star and (b) that arising from small aspherical perturbations. The latter can result from large-scale motion, such as giant convective cells, or, in a magnetic rotating star, from the neglected small-body force $\mathcal{F}$ in eq. (1.2.1) and from advection due to rotation of the basic state. The first kind of perturbation is relevant to the study of small frequency differences between two similar stars and of frequency differences resulting from small changes in the structure of a given star, such as may occur during a stellar cycle. In the case of the solar cycle, the changes might have arisen from the spherically symmetric component of a magnetic perturbation, $\mathcal{F}$. Perturbations of type (a) are also important to helioseismology and asteroseismology, in which small differences between the observed frequencies of a star and the eigenfrequencies in a theoretical model of that star are used to estimate the structural differences between the star and the model.

Although perturbations of type (b) may have a spherically symmetrical component, the main interest lies in their symmetry breaking, which induces a splitting of the degeneracy of the eigenfrequencies with respect to the azimuthal order m of the modes.

7.1. Spherically symmetric perturbations

My starting point is the variational principle (5.3.1)–(5.3.5). Suppose the difference between two equilibrium states is denoted by Δp, $\Delta\rho$ and $\Delta\gamma$, and the corresponding changes in the eigenfunctions and their associated eigenfrequencies by $\Delta\xi$ and $\Delta\omega$, where, when referring to functions, Δ denotes the difference between the two at a fixed position in space. Then, provided the changes are small, in the sense that $|\Delta p/p| \ll 1$, etc. everywhere in the star, the two integral relations (5.3.1) corresponding to the two states can be subtracted and the resulting equation linearized in the perturbation quantities. The expression thus obtained for the frequency perturbation, $\Delta\omega$, depends formally on two kinds of terms: those depending directly on the differences Δp, $\Delta\rho$ and $\Delta\gamma$ between the two equilibrium states and those depending indirectly on these differences through $\Delta\xi$. However, since eq. (5.3.1) is stationary with respect to variations in ξ that satisfy the boundary conditions of the problem, the terms containing $\Delta\xi$ cancel. If all one is interested in is frequencies, this allows a very substantial simplification, since this requires one to know the eigenfunctions and eigenfrequencies of only one of the two states; it is not necessary to solve the appropriately perturbed equations (5.1.4)–(5.1.7) or (5.1.8)–(5.1.11).

It is important to realize that for eq. (5.3.1) to be valid, the equilibrium state must satisfy the equations of hydrostatic support (2.1)–(2.3). Unless the equilibrium state is static on a dynamical time scale, the decomposition (3.12) is not valid, and the concept of frequency is not well-defined. However, because of the large disparity between the dynamical and thermal time scales, it is hardly necessary for the star to be in thermal balance.

In connection with this point it is also worth remarking that adiabatic oscillations of a star do not depend directly on temperature. They depend basically on the momentum equation, which balances the rate of change of momentum against pressure gradients. This relates inertia-density, ρ, and frequency, via acceleration, to pressure p. It is also necessary, of course, to know how p responds to changes in ρ, and this is provided by the constitutive relation $\delta\ln p = \gamma\,\delta\ln\rho$ for small adiabatic changes. This fundamental dependence is quite evident from the fact that p, ρ and γ are the only functions of the equilibrium state upon which the adiabatic pulsation equations (3.6)–(3.11) depend. It follows, therefore, that frequency differences depend only on differences in p, ρ and γ. Conversely, it is not possible to learn anything other than the stratification of p, ρ and γ from a knowledge of only the frequencies of oscillation of a nonmagnetic, nonrotating star. To make deductions about the temperature, or any other derived thermodynamic quantity, requires the utilization of the equation of state, which cannot be accomplished without knowledge of at least the chemical composition of the stellar material.

It is straightforward to perturb eqs. (5.3.1)–(5.3.5). For simplicity I assume ω to be well below the critical acoustic frequency, ω_c, in the atmosphere, so that the boundary terms, such as $\mathcal{B}$, can be ignored. I also assume, again for simplicity, that the two equilibrium states have the same mass and radius; it is straightforward to retain the additional terms when this is not the case. Then, since perturbations to $\boldsymbol{\xi}$ do not contribute to first order,

$$\begin{aligned}\mathcal{I}\Delta\omega^2 = & -\omega^2\int_0^R(\xi^2+\eta^2)r^2\Delta\rho\,\mathrm{d}r + \int_0^R p\chi^2r^2\Delta\gamma\,\mathrm{d}r \\ & + \int_0^R\left[\gamma\chi^2\Delta p + 2\xi\left(\chi+\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\xi\right)\frac{\mathrm{d}\Delta p}{\mathrm{d}r} - g\rho\xi^2\frac{\mathrm{d}\,\Delta\ln\rho}{\mathrm{d}r}\right]r^2\,\mathrm{d}r \\ & -\Delta W, \end{aligned} \tag{7.1.1}$$

where ξ and η are the vertical and horizontal components of the displacement amplitude defined by eq. (5.1.1), χ represents the amplitude of the dilatation:

$$\chi = r^{-2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi) - \frac{L}{r}\eta, \tag{7.1.2}$$

where $L^2 = l(l+1)$, and I have used the quantity

$$\mathcal{I} = \int_0^R(\xi^2+\eta^2)\rho r^2\,\mathrm{d}r, \tag{7.1.3}$$

which is proportional to the measure $I(\boldsymbol{\xi},\boldsymbol{\xi})$ of the modal inertia, defined by eq. (5.3.3). The quantity W is the perturbation to the gravitational energy associated with the oscillations:

$$W = -\int_0^R \Phi'\,\mathrm{div}\,\rho\boldsymbol{\xi}\,r^2\,\mathrm{d}r,$$

and is equal to $1/4\pi$ of the magnitude of the second of the two terms on the right-hand side of eq. (5.3.2). It can be conveniently rewritten in terms of the amplitude $\rho' = -\mathrm{div}\,\rho\boldsymbol{\xi} = -\rho\chi - \xi\,\mathrm{d}\rho/\mathrm{d}r$ of the Eulerian density fluctuation by using eq. (3.10) and substituting for Φ' that solution of eq. (5.1.7), with $-4\pi G\rho'$ on the right-hand side, which is regular both at the origin and at infinity:

$$W = \frac{8\pi G}{2l+1}\int_0^R r^{-l+1}\rho'(r)\int_0^r s^{l+1}\rho'(s)\,\mathrm{d}s\,\mathrm{d}r. \tag{7.1.4}$$

(Here, as in eq. (7.1.3), the prime denotes Eulerian perturbation due to the oscillations.) On perturbing at constant $\boldsymbol{\xi}$ the equilibrium state implicit in eq. (7.1.4), ΔW is obtained in terms of $\Delta\rho(r)$ and its first derivative.

Since both basic states are in hydrostatic equilibrium, the perturbations Δp and $\Delta\rho$, defining their difference, are not independent. Instead, they are related by the perturbed hydrostatic equations (2.1)–(2.3):

$$\frac{\mathrm{d}\Delta p}{\mathrm{d}r} = -\frac{G}{r^2}(\rho\Delta m + m\Delta\rho), \tag{7.1.5}$$

where

$$\Delta m(r) = 4\pi\int_0^r s^2\Delta\rho(s)\,\mathrm{d}s, \tag{7.1.6}$$

from which follows that for any regular function $f(r)$,

$$\begin{aligned}\int_0^R f(r)\frac{\mathrm{d}\Delta p}{\mathrm{d}r}\,\mathrm{d}r & = -G\int_0^R r^{-2}f(r)\left[4\pi\rho(r)\int_0^r s^2\Delta\rho(s)\,\mathrm{d}s + m(r)\Delta\rho(r)\right]\mathrm{d}r \\ & = \int_0^R F\Delta\rho\,\mathrm{d}r, \end{aligned} \tag{7.1.7}$$

where

$$F(r) = -G\left[4\pi r^2\int_r^R s^{-2}\rho(s)f(s)\,\mathrm{d}s + r^{-2}m(r)f(r)\right]. \tag{7.1.8}$$

Thus the term in eq. (7.1.1) involving $\mathrm{d}\Delta p/\mathrm{d}r$ can be written in terms of $\Delta\rho$. After integration by parts, the term involving the undifferentiated Δp can be written in terms of $\mathrm{d}\Delta p/\mathrm{d}r$, and hence in terms of $\Delta\rho$. The term involving $\mathrm{d}\Delta\rho/\mathrm{d}r$ is also integrated by parts. The final result can thus be written in the form

$$\frac{\Delta\omega}{\omega} = \int_0^R(\mathcal{K}_{\gamma,\rho}\,\Delta\ln\gamma + \mathcal{K}_{\rho,\gamma}\,\Delta\ln\rho)\,\mathrm{d}r, \tag{7.1.9}$$

where

$$\mathcal{K}_{\gamma,\rho} = \tfrac{1}{2}\omega^{-2}\mathcal{I}^{-1}\gamma p r^2(\mathrm{div}\,\boldsymbol{\xi})^2, \tag{7.1.10}$$

and, after some manipulation (and using the assumption that both equilibrium states have the same mass and radius, so that $\Delta m(R) = 0$),

$$2\omega^2 \mathcal{I} \mathcal{K}_{\rho,\gamma} = 4\pi G \rho r^2 \int_r^R \psi(s)\,\mathrm{d}s + Gm\rho\psi(r) + G\frac{\mathrm{d}}{\mathrm{d}r}(m\rho\xi^2) - \omega^2\rho r^2(\xi^2+\eta^2) + \frac{8\pi G\rho}{2l+1}\left\{(l+1)r^{-l}(\xi-\eta)\int_0^r s^{l+2}\rho'(s)\,\mathrm{d}s - lr^{l+1}[\xi+(l+1)\eta]\int_r^R s^{-l+1}\rho'(s)\,\mathrm{d}s\right\}, \qquad (7.1.11)$$

where

$$\psi(r) = r^{-2}\int_0^r \gamma s^2\chi^2\,\mathrm{d}s - \left(2\chi + \xi\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right)\xi. \qquad (7.1.12)$$

Equation (7.1.9) can be used for inferring the hydrostatic stratification of the Sun from measurements of solar oscillation eigenfrequencies. If $\Delta\omega^2$ represents the difference between observed and theoretical frequencies, the coupled integral constraints can be used to estimate $\Delta\ln\gamma$ and $\Delta\ln p$, from which an improved estimate of the solar structure can be obtained. Methods by which that might be accomplished are reviewed by Gough (1985). The procedure can be repeated until, hopefully, the iterations converge.

If, on the other hand, one wanted to compute frequency changes resulting from, say, a thermal redistribution in a star, the relation between $\Delta\gamma$, Δp and $\Delta\rho$ might then be known. In that case $\Delta\ln\gamma$ can be eliminated from eq. (7.1.9); $\mathcal{K}_{\gamma,\rho}$ is formally set to zero and the consequent modifications to $\mathcal{K}_{\rho,\gamma}$ are adding to the right-hand side of eq. (7.1.11) the term $\gamma p r^2(\mathrm{div}\,\boldsymbol{\xi})^2\gamma_\rho$ and replacing γ in eq. (7.1.12) by $\gamma(1+\gamma_p)$, where γ_ρ and γ_p are the partial logarithmic derivatives of γ with respect to ρ at constant p and to p at constant ρ, respectively. In addition there will be a contribution to $\Delta\ln\omega$ from any spatial redistribution of the hydrogen abundance, X (which itself would be expressible in terms of $\Delta\ln\rho$ if the Lagrangian variation were zero), the kernel for $\Delta\ln X$ being $\gamma p r^2(\mathrm{div}\,\boldsymbol{\xi})^2\gamma_X$, in an obvious notation.

It is possible to recast eq. (7.1.9) in terms of other pairs of perturbation variables with the help of eqs. (7.1.5) and (7.1.6), or, equivalently, transformations such as eqs. (7.1.7), (7.1.8). For example, since high-frequency p modes depend mainly on the sound speed, c, it might be considered more useful to use c^2 and, say, a variable, such as γ, that one expects not to vary a great deal. This can be achieved by transforming just sufficient of $\mathcal{K}_{\rho,\gamma}\,\Delta\ln\rho$ in eq. (7.1.9) back to a term in $\Delta\ln p$ to make it possible to write the outcome in terms of $\Delta\ln c^2$, using the relation

$$\Delta\ln c^2 = \Delta\ln p - \Delta\ln\rho + \Delta\ln\gamma. \qquad (7.1.13)$$

Specifically, one can set $\mathcal{K}_{\rho,\gamma} = \Phi_1 - \Phi_2$, choosing the separation into two terms in such a way that

$$\int_0^R \Phi_1\,\Delta\ln\rho\,\mathrm{d}r = \int_0^R \Phi_2\,\Delta\ln p\,\mathrm{d}r. \qquad (7.1.14)$$

Then

$$\frac{\Delta\omega}{\omega} = \int_0^R (\mathcal{K}_{\gamma,c^2}\,\Delta\ln\gamma + \mathcal{K}_{c^2,\gamma}\,\Delta\ln c^2)\,\mathrm{d}r, \qquad (7.1.15)$$

where $\mathcal{K}_{c^2,\gamma} = \Phi_2$ and $\mathcal{K}_{\gamma,c^2} = \mathcal{K}_{\gamma,\rho} - \Phi_2$. With the help of the hydrostatic constraints (7.1.5) and (7.1.6) and some integrations by parts, taking care that the integrated terms vanish (which is assured at $r = 0$ by the behaviour of the eigenfunctions discussed in sections 4.2 and 5.2), one can show that in order for condition (7.1.14) to be satisfied, Φ_2 must be chosen to be $p\,\mathrm{d}\Phi/\mathrm{d}r$, where

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p}{r^2\rho}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Gm\Phi}{r^4}\right) + \frac{4\pi G\rho}{r^2}\Phi = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p}{r^4}\mathcal{K}_{\rho,\gamma}\right). \qquad (7.1.16)$$

Thus one finds, as indeed is usually the case when transforming the perturbation variables, that the new kernels cannot be written in closed form, but are instead expressed in terms of the solution of a second-order, linear, inhomogeneous differential equation, the inhomogeneous term containing one of or both the old kernels. In general this equation must be solved numerically. However, in the special case of a plane-parallel envelope model in the Cowling approximation, e.g., which is a good approximation for high-degree p modes, it is possible to express the kernels $\mathcal{K}_{\gamma,c^2}$ and $\mathcal{K}_{c^2,\gamma}$ in closed form (Gough and Toomre 1983). It is a very simple matter to cast $\Delta\omega^2$ in terms of $\Delta\ln\rho$ and $\Delta\ln c^2$ directly from eq. (7.1.1).

For asymptotic modes, the forms (5.8.7) and (5.8.8) or (5.8.9) and (5.8.10) can be substituted into formulae such as eqs. (7.1.10)–(7.1.12) for the kernels. However, it is simpler to perturb the eigenvalue equation (5.8.1) directly. For high-order p modes one can approximate eq. (5.8.1) by eq. (5.8.21) with the second integral neglected; a sound-speed perturbation Δc produces a frequency shift

$$\frac{\Delta\omega}{\omega} \simeq \int_{r_1}^R \mathcal{K}\frac{\Delta c}{c}\,\mathrm{d}r, \qquad (7.1.17)$$

where

$$\mathcal{K} = S^{-1}c^{-1}(1 - a^2/w^2)^{-1/2}, \qquad (7.1.18)$$

$$S = \int_{r_1}^{R} c^{-1}(1 - a^2/w^2)^{-1/2}\,dr, \tag{7.1.19}$$

and r_1 satisfies $\omega r_1 = Lc(r_1)$. In deriving this formula I have ignored the small contribution from the variation in α. The quantity S is proportional to asymptotic approximations to the measures $I(\boldsymbol{\xi}, \boldsymbol{\xi})$, defined by eq. (5.3.3), and $\mathcal{I}$, defined by eq. (7.1.3), of the inertia of the mode.

7.2. Aspherical scalar perturbations: degenerate perturbation theory

The variational principle (5.3.1)–(5.3.5) does not assume the star to be spherically symmetrical. Therefore, if the star is essentially in hydrostatic equilibrium but deviates from being spherically symmetrical by a small amount, the perturbations to the eigenfrequencies can still be calculated by the same procedure. Some care must be taken, however, to ensure that the eigenfunctions, $\boldsymbol{\xi}$, satisfy the boundary conditions in the perturbed model. This can easily be accomplished by introducing a distorted radial coordinate

$$x = [1 + h(r, \theta, \phi)]r, \quad |h| \ll 1, \tag{7.2.1}$$

such that $x = 1$ on the perturbed surface of the star. Then, if $\boldsymbol{\xi} =: (\xi, \eta)$ is regarded as a function of x, rather than r, in the perturbed model, eq. (5.3.1) can be perturbed at constant $\boldsymbol{\xi}$. In principle, for infinitesimal perturbations, one has the freedom to choose h at will, provided it is such that the linearized equation for the surface of the star is $r = (1 - h)R$. In practice, however, for small, finite perturbations, it is prudent to choose h such that it follows in some sense the distortion of regions of rapid variation, such as ionization zones, since then ignoring the variation of $\boldsymbol{\xi}$ introduces a smaller error than it otherwise might do.

The analysis is carried through in much the same way as for spherically symmetrical perturbations, although now the angular integrals introduce geometrical factors causing $\Delta \ln \gamma$, $\Delta \ln \rho$ and $\Delta \ln c^2$ in eqs. (7.1.9) and (7.1.15) to be replaced by

$$\bar{f} := \frac{\int_0^{2\pi} d\phi \int_{-1}^{1} f(r, \mu, \phi)[Y_l(\mu, \phi)]^2\,d\mu}{\int_0^{2\pi} d\phi \int_{-1}^{1} [Y_l(\mu, \phi)]^2\,d\mu}, \tag{7.2.2}$$

where $\mu = \cos\theta$, Y_l is a spherical harmonic of degree l and is an appropriate linear combination (which I will determine below) of functions $P_l^m(\cos\theta)\,e^{im\phi}$ of the same degree l and different orders m, determining the angular dependence of the eigenfunction, and $f = \Delta \ln \gamma$, $\Delta \ln \rho$ or $\Delta \ln c^2$ as appropriate, the perturbations to γ, ρ and c^2 now being computed at constant x. This can be seen immediately from an inspection of the structure (5.1.1) of $\boldsymbol{\xi}$ and the forms of K and I, defined by eqs. (5.3.2) and (5.3.3), on noting that if the radial component of $\boldsymbol{\xi}$ is proportional to $Y_l(\mu, \phi)$ then so is div $\boldsymbol{\xi}$, and that $\boldsymbol{\xi}^* \cdot \boldsymbol{\xi} \propto [Y_l(\mu, \phi)]^2$ + terms which integrate to zero.

One consequence of the symmetry breaking is that the modes represented by the expressions (5.1.1) and (5.1.2) are no longer degenerate. The different unperturbed eigenfunctions corresponding to different values of m or to different coordinate axes, e.g., are no longer equivalent, and only to the members of a particular set of the eigenfunctions can one ascribe, to leading order in the perturbation, single frequencies. The nature of the aspherical perturbation to the basic state determines such a set. Of course any eigenfunction of degree l of the unperturbed spherically symmetrical state can be written as a linear combination of any $2l+1$ independent eigenfunctions of degree l and the same order n. Here I take

$$\boldsymbol{\xi}_l^k = \sum_m \left[\left(l + \frac{1}{2}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} c_m \boldsymbol{\xi}_{lm}, \tag{7.2.3}$$

where each $\boldsymbol{\xi}_{lm}$ is of the form given by eq. (5.1.1), the scaling factors in front of the coefficients c_m having been chosen to normalize the spherical harmonics such as to have an rms value of unity over the surface of a sphere. The superscript k is a label that identifies one of the $2l+1$ possible independent linear combinations. It is omitted from c_m to simplify the notation. Note that the summation convention is not being used in this discussion.

The coefficients c_m are determined by the form of the perturbations f. It is convenient to expand the angular dependence of these perturbations in spherical harmonics:

$$f(r, \mu, \phi) = \sum_{\lambda > 0} \sum_{m=-\lambda}^{\lambda} \psi_{f\lambda}^m(r) P_\lambda^m(\mu)\,e^{im\phi}, \tag{7.2.4}$$

where $\psi_{f\lambda}^m$ is real and $(\lambda - m)!\,\psi_{f\lambda}^{-m} = (\lambda + m)!\,\psi_{f\lambda}^m$, ensuring the reality of f. Then the analogue of eq. (7.1.9) is

$$\sum_m \sum_{m'} c_m c_{m'} \times \left[\sum_\lambda Q_{\lambda l}^{mm'} \int_0^R (\mathcal{K}_{\gamma,\rho}\psi_{\gamma\lambda}^{m''} + \mathcal{K}_{\rho,\gamma}\psi_{\rho\lambda}^{m''})\,dr - \frac{\Delta\omega_{nl}^k}{\omega_{nl}}\delta_{mm'}\right] = 0, \tag{7.2.5}$$

where

$$Q_{\lambda l}^{mm'} = \left(l + \frac{1}{2}\right)\left[\frac{(l-m)!(l-m')!}{(l+m)!(l+m')!}\right]^{1/2} \int_{-1}^{1} P_l^m(\mu) P_l^{m'}(\mu) P_\lambda^{m''}(\mu)\,d\mu, \tag{7.2.6}$$

$$m - m' \pm m'' = 0 \tag{7.2.7}$$

and δ_{ik} is the Kronecker delta. Since the expression on the left-hand side of eq. (7.2.5) is stationary with respect to variations in the eigenfunctions, and hence variations in the coefficients c_m, it follows that the coefficients are the eigenvectors of the matrix equation

$$\sum_{m'} \left(A_{mm'} - \frac{\Delta\omega_{nl}^k}{\omega_{nl}} \delta_{mm'} \right) c_{m'} = 0, \tag{7.2.8}$$

where

$$\begin{aligned} A_{mm'} &= \sum_\lambda Q_{\lambda l}^{mm'} \int_0^R [\mathcal{K}_{\gamma,\rho} \psi_{\gamma\lambda}^{\mp(m-m')} + \mathcal{K}_{\rho,\gamma} \psi_{\rho\lambda}^{\mp(m-m')}]\, \mathrm{d}r \\ &=: \sum_\lambda a_\lambda Q_{\lambda l}^{mm'}, \end{aligned} \tag{7.2.9}$$

and that the relative perturbations $\Delta\omega/\omega$ are its eigenvalues. The resulting set of eigenfrequencies of modes with the same l and n is often called a *multiplet*, each member of which is a *singlet*.

The geometrical factors $Q_{\lambda l}^{mm'}$ are related to Clebsch–Gordan coefficients. Their symmetry properties are discussed, e.g., by Edmonds (1957). The first property to which I wish to draw attention here, which is immediately evident by inspection of the definition (7.2.6), is that if m and m' are replaced by $-m$ and $-m'$, $Q_{\lambda l}^{mm'}$ is unchanged. The scalar perturbations f have the same effect on eastward and westward propagating waves that are otherwise identical. Consequently, the $2l+1$ eigenvalues $\Delta\omega_{nl}^k$ are degenerate in pairs, whatever the order of the mode. In general, therefore, there are only $l+1$ distinct eigenfrequencies; of course, for particular perturbations $\psi_{\gamma\lambda}^m(r)$, $\psi_{\rho\lambda}^m(r)$ there may be additional accidental coincidences, but these would occur only for specific modes rather than for all orders n. A second property is that $Q_{\lambda l}^{mm'} = 0$ if λ is odd; this follows from the reflectional symmetry properties of P_l^m and the selection rule (7.2.7). Consequently, only about half of the terms in the expansion (7.2.4) contribute in leading order to the frequency splitting.

In the special case of axisymmetric perturbations to the basic state the problem simplifies considerably. If one takes the coordinate axis to be the axis of symmetry, then $\psi_{f\lambda}^m = 0$ if $m \neq 0$, and the matrix $A_{mm'}$ is diagonal. The solution of eq. (7.2.8) for mode k is simply $c_m \propto \delta_{mk}$, which means that the basis functions ξ_{lm} are the leading-order eigenfunctions. The frequency perturbations are given by

$$\frac{\Delta\omega_{nlm}}{\omega_{nl}} = A_{mm'} = \sum_\lambda Q_{\lambda lm} \int_0^R (\mathcal{K}_{\gamma,\rho} \psi_{\gamma\lambda}^0 + \mathcal{K}_{\rho,\gamma} \psi_{\rho\lambda}^0)\, \mathrm{d}r, \tag{7.2.10}$$

where $Q_{\lambda lm} := Q_{\lambda l}^{mm}$; I have lowered the mode label in my notation for the frequency perturbation to emphasize that it now meaningfully denotes azimuthal order m. It is interesting to note that

$$\sum_{m=-l}^{l} Q_{\lambda lm} = 0 \quad \text{for } \lambda > 0, \tag{7.2.11}$$

so the mean frequency of a multiplet is equal to the frequency of the degenerate modes of the spherically symmetrical unperturbed state. This result is also true for nonaxisymmetrical perturbations to the equilibrium state, since

$$\sum_k \Delta\omega_{nl}^k = \omega_{nl}\, \mathrm{trace}(A) = \sum_{\lambda>0} a_\lambda \sum_{m=-l}^{l} Q_{\lambda l}^{mm} = 0. \tag{7.2.12}$$

When $\lambda = 0$ and $Q_{\lambda lm} = 1$, the spherically symmetric contribution to the perturbation to the equilibrium state causes the frequencies of all the modes with the same n and l to be shifted by the same amount, and does not contribute to the degeneracy splitting.

For the purpose of comparing the results of this expansion in the axisymmetric case with the asymptotic discussion of section 8, it will be convenient to separate $P_\lambda(\mu)$ into its powers of μ. Thus, e.g., for a sound-speed perturbation

$$\frac{\Delta c}{c} = \sum_\nu c_\nu(r) \mu^{2\nu} \tag{7.2.13}$$

(the coefficients $c_\nu(r)$ being different from the constants c_m of eq. (7.2.3)), the frequency perturbation $\Delta\omega$ of a high-order p mode with degree l and azimuthal order m satisfies

$$\frac{\Delta\omega}{\omega} \simeq \sum_\nu \tilde{Q}_{\nu lm} \int_{r_1}^R \mathcal{K} c_\nu \, \mathrm{d}r, \tag{7.2.14}$$

with $\mathcal{K}$ defined by eq. (7.1.18) and where

$$\tilde{Q}_{\nu lm} = \left(l + \frac{1}{2} \right) \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} \mu^{2\nu} [P_l^m(\mu)]^2 \, \mathrm{d}\mu. \tag{7.2.15}$$

From using the recurrence relation

$$(2l+1)\mu P_l^m = (l+1-m) P_{l+1}^m + (l+m) P_{l-1}^m \tag{7.2.16}$$

follows that

$$\tilde{Q}_{1lm} = R^m_{l+1} + R^m_l, \tag{7.2.17}$$

$$\tilde{Q}_{2lm} = S^m_{l+1} + S^m_l, \quad \text{etc.,} \tag{7.2.18}$$

where

$$R^m_l = \frac{l^2 - m^2}{4l^2 - 1}, \qquad S^m_l = R^m_l (R^m_{l-1} + R^m_l + R^m_{l+1}), \quad \text{etc.} \tag{7.2.19}$$

Since I am concentrating now on asymptotics for which the wavelength of the wave is much smaller than the scale of variation of the equilibrium state, including that of the aspherical perturbation, only the case $l \gg \nu$ is relevant. Then

$$R^m_l \sim \frac{1}{4}\left(1 - \frac{m^2}{L^2}\right). \tag{7.2.20}$$

Using this approximation in the recurrence relations (7.2.17)–(7.2.19) yields

$$\tilde{Q}_{\nu lm} \sim \frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\left(1 - \frac{m^2}{L^2}\right)^\nu. \tag{7.2.21}$$

This relation is rederived by a direct asymptotic expansion of the wave equation in section 8. From the expansions of the Legendre polynomial

$$P_{2\nu}(\cos\theta) = \frac{1}{2^{2\nu}}\sum_{k=0}^{\nu}(-1)^k \frac{(4\nu - 2k)!}{k!(2\nu - k)!(2\nu - 2k)!}\cos^{2(\nu-k)}\theta \tag{7.2.22}$$

$$= \frac{(\nu!)^2}{(2\nu)!}\sum_{k=0}^{\nu}(-1)^{\nu-k}\frac{(4\nu - 2k)!}{2^{2(\nu-k)}k!(2\nu - k)![(\nu - k)!]^2}\sin^{2(\nu-k)}\theta, \tag{7.2.23}$$

follows immediately that, when $l \gg \lambda$,

$$Q_{2\lambda lm} \sim \frac{(-1)^\lambda (2\lambda)!}{2^{2\lambda}(\lambda!)^2} P_{2\lambda}(m/L). \tag{7.2.24}$$

7.3. *Advection by rotation*

The introduction of slow rotation $\boldsymbol{\Omega}(r,\theta) = \Omega(r,\theta)\,\boldsymbol{k}$ about a unique axis defined by the constant unit vector $\boldsymbol{k}$ (which is not to be confused with the wave number elsewhere) also breaks the symmetry and splits the degeneracy of the eigenfrequencies. There are two contributions. One comes from the extra advection terms in the linearized adiabatic oscillation equation (3.6). The other comes from the distortion of the equilibrium state by the centrifugal force. The latter is simply a scalar perturbation to p, ρ and γ, and can be dealt with by the method outlined in the previous subsection.

To keep the discussion simple, I restrict attention to correction terms that are linear in Ω/ω. These arise only from the advection terms in the equations of motion. As I will discuss later, if the star contains a large-scale rigidly rotating magnetic field it is convenient to transform to a coordinate system rotating with the field. In that case the Coriolis force must also be included. The effects of the centrifugal force on both the equilibrium state and the oscillation dynamics are quadratic in Ω/ω, and will be ignored here; if they were included it would be necessary to include also the perturbed advection terms and the first-order correction to the eigenfunctions. This is discussed by Gough and Thompson (1990).

To evaluate the perturbation due to any steady velocity field $\boldsymbol{U}(\boldsymbol{r})$ in the equilibrium state one simply replaces $\boldsymbol{u}$ by $\boldsymbol{U} + \boldsymbol{u}$ in the adiabatic equations of motion (1.2.1)–(1.2.5), and linearizes them to obtain the analogue of eqs. (3.6)–(3.11). I will restrict attention to the case of pure rotation, where $\boldsymbol{U} = \Omega\boldsymbol{k} \times \boldsymbol{r}$.

An integral relation for the eigenfrequency ω can be obtained by following the procedure of section 5.3. This relation is

$$I(\boldsymbol{\xi}, \boldsymbol{\xi}^*)\omega^2 - 2\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)\omega - K(\boldsymbol{\xi}, \boldsymbol{\xi}^*) + \mathcal{B}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = 0, \tag{7.3.1}$$

where K, I and $\mathcal{B}$ are defined by eqs. (5.3.2)–(5.3.4) and

$$\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) := \mathrm{i}\int_{\mathcal{V}} \boldsymbol{\xi}^* \cdot (\boldsymbol{U} \cdot \nabla\boldsymbol{\xi})\,\rho\,\mathrm{d}V. \tag{7.3.2}$$

It was pointed out in section 5.3 that in the absence of $\mathcal{R}$ eq. (7.3.1) is a variational principle. Actually, in view of the symmetry properties of $\mathcal{R}$, it is straightforward to demonstrate that it remains a variational principle when $\mathcal{R}$ is retained (it is a special case of the more general variational principle derived by Lynden-Bell and Ostriker (1967)), though that stronger property is not needed here. It is adequate to note simply that the modifications to the eigenfunctions of the nonrotating star make no first-order contribution to $(K - \mathcal{B})/I$, implying that the correction to the eigenfrequency to leading order in Ω/ω is $\omega_\Omega = \mathcal{R}/I$, where $\mathcal{R}$ and I are evaluated with the appropriate eigenfunctions of the nonrotating state.

If $\boldsymbol{k}$ is chosen to be the axis of spherical polar coordinates, then

$$\mathcal{R}(\boldsymbol{\xi}_{lm}, \boldsymbol{\xi}^*_{lm'}) = 0, \quad \text{if } m \neq m', \tag{7.3.3}$$

where, as in the expansion (7.2.3), ξ_{lm} is given by eq. (5.1.1). Consequently, the leading-order terms of the perturbed nondegenerate eigenfunctions are the functions ξ_{lm}. The azimuthal order m can therefore be retained to label the modes, and eq. (7.3.2) can be rewritten:

$$\mathcal{R}(\xi_{lm}, \xi^*_{lm}) = \int_{\mathcal{V}} [m\xi \cdot \xi^* + \mathrm{i}k \cdot (\xi \times \xi^*)]\Omega\rho\, \mathrm{d}V. \tag{7.3.4}$$

If the angular velocity, Ω, is expanded in even powers of $\mu = \cos\theta$, thus

$$\Omega(r,\theta) = \sum_\lambda \Omega_\lambda(r)\mu^{2\lambda} \tag{7.3.5}$$

(as in the case of scalar perturbations, there is no contribution to $\mathcal{R}$ from odd functions of μ), the frequency perturbation $\Delta\omega_{nlm}$ is given by

$$\Delta\omega_{nlm} = m\mathcal{I}^{-1}\sum_\lambda \int_0^R (\{\xi^2 - 2L^{-1}\xi\eta + [1 - L^{-2}(\lambda+1)(2\lambda+1)]\eta^2\}\tilde{Q}_{\lambda lm} + L^{-2}\lambda(2\lambda-1)\eta^2\tilde{Q}_{\lambda-1\, lm})\Omega_\lambda\rho r^2\, \mathrm{d}r, \tag{7.3.6}$$

$$\mathcal{I} = \int_0^R (\xi^2 + \eta^2)\rho r^2\, \mathrm{d}r. \tag{7.3.7}$$

From $\tilde{Q}_{\lambda l\, -m} = \tilde{Q}_{\lambda lm}$ follows that

$$\Delta\omega_{nl\, -m} = -\Delta\omega_{nlm} \tag{7.3.8}$$

and, of course,

$$\sum_{m=-\lambda}^{\lambda} \Delta\omega_{nlm} = 0. \tag{7.3.9}$$

In the case of spherically symmetrical rotation ($\Omega_\lambda = 0$ if $\lambda \neq 0$)

$$\Delta\omega_{nlm} = m\mathcal{I}^{-1}\int_0^R (\xi^2 + \eta^2 - 2L^{-1}\xi\eta - L^{-2}\eta^2)\Omega\rho r^2\, \mathrm{d}r; \tag{7.3.10}$$

the perturbed frequency is proportional to m. A direct consequence of this result is that in a frame rotating with angular velocity $m^{-1}\Delta\omega_{nlm}$ degeneracy is restored. Another way of stating the result is that the effect of a spherically symmetrical

angular velocity $\Omega(r)$, to first order in Ω/ω, is simply to rotate rigidly, with angular velocity $m^{-1}\Delta\omega_{nlm}$, each wave pattern formed by a superposition of modes with the same n and l. In the case when Ω is independent of r, that angular velocity is $(1 - C)\Omega$, where

$$C = \mathcal{I}^{-1}\int_0^R L^{-1}(2\xi + L^{-1}\eta)\eta\rho r^2\, \mathrm{d}r, \tag{7.3.11}$$

a result first derived by Cowling and Newing (1948) and Ledoux (1951).

It is useful to record the rotational splitting formulae for asymptotic modes, which are obtained by substituting the expressions (5.8.7), (5.8.8) or (5.8.9), (5.8.10) into eq. (7.3.6). Of course, strictly speaking these forms are valid only far from the turning points; nevertheless, despite the formal divergences of the integrands when these forms are used, the contributions from the neighbourhoods of the turning points are finite, and actually are quite small. Therefore the resulting expressions are quite good approximations. For p modes I simplify the formula by neglecting ω_c^2/Ω^2 and $(LcN/r\omega^2)^2$ compared with unity in the expression (5.4.8) for K and by approximating u^2 by $\omega^2 r^{-2}\rho$, yielding

$$\frac{\Delta\omega_{nlm}}{m} \sim \frac{\sum_\lambda \int_{r_1}^R (\tilde{Q}_{\lambda lm} + aw^{-1}\tilde{T}_{\lambda lm})(1 - a^2/w^2)^{-1/2}c^{-1}\Omega_\lambda\, \mathrm{d}r}{\int_{r_1}^R (1 - a^2/w^2)^{-1/2}c^{-1}\, \mathrm{d}r}, \tag{7.3.12}$$

where

$$\tilde{T}_{\lambda lm} = L^{-2}[\lambda(2\lambda-1)\tilde{Q}_{\lambda-1\, lm} - (\lambda+1)(2\lambda+1)\tilde{Q}_{\lambda lm}]. \tag{7.3.13}$$

Since $a/w \ll 1$ well away from the lower turning point, provided λ is not large compared with L the contribution containing $\tilde{T}_{\lambda lm}$ is much less than that containing $\tilde{Q}_{\lambda lm}$. Therefore the formula may be simplified still further:

$$\frac{\Delta\omega_{nlm}}{m} \simeq \sum_\lambda \tilde{Q}_{\lambda lm}\int_{r_1}^R \mathcal{K}\Omega_\lambda\, \mathrm{d}r, \tag{7.3.14}$$

where $\mathcal{K}(r)$ is given by eq. (7.1.18). For g modes, $(\omega^2 - \omega_c^2)r^2/L^2c^2$ is neglected compared with unity in the definition of K and u^2 is approximated by $-L^2\omega^{-2}r^{-4}g^2\rho$, resulting in

$$\frac{\Delta\omega_{nlm}}{m} \sim \sum_\lambda \left(\tilde{Q}_{\lambda lm}\int_{r_1}^{r_2} \mathcal{K}^-\Omega_\lambda\, \mathrm{d}r + \tilde{T}_{\lambda lm}\int_{r_1}^{r_2} \mathcal{K}^+\Omega_\lambda\, \mathrm{d}r\right), \tag{7.3.15}$$

where

$$\mathcal{K}^{\pm}(r) = \frac{(1-\omega^2/\mathcal{N}^2)^{\pm 1/2} r^{-1}\mathcal{N}}{\int_{r_1}^{r_2}(1-\omega^2/\mathcal{N}^2)^{-1/2} r^{-1}\mathcal{N}\,\mathrm{d}r}\,. \tag{7.3.16}$$

Except when l is large, the two contributions for each value of λ are comparable.

7.4. *Internal magnetic field*

As in the case of rotation, a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ influences the oscillations both directly, through the introduction of a (perturbed) Lorentz force in the oscillation momentum equation, and indirectly via the modification of the equilibrium state by the unperturbed Lorentz force.

I will restrict attention to magnetic fields that do not penetrate the surface, $\mathcal{S}$, of the star and which produce stresses that are everywhere much less than the gas-pressure gradient. In that case nonsingular (degenerate) perturbation theory can be applied. (If a significant field were to penetrate the surface, there would be a region where the Lorentz force is potentially large, producing a singular perturbation. That is much more difficult to deal with (e.g. Goossens et al. 1976, Biront et al. 1982, Roberts and Soward 1983, Campbell and Papaloizou 1986).) The analogue of eq. (7.3.1) is then

$$(I+I_B)\omega^2 - (K+K_B) - 2\mathcal{M} + \mathcal{B} = 0, \tag{7.4.1}$$

where

$$2\mathcal{M} := -\mu_0^{-1}\int_{\mathcal{V}} \boldsymbol{\xi}^*\cdot[(\rho^{-1}\operatorname{div}\rho\boldsymbol{\xi})(\operatorname{curl}\boldsymbol{B})\times\boldsymbol{B} + (\operatorname{curl}\boldsymbol{B}')\times\boldsymbol{B} + (\operatorname{curl}\boldsymbol{B})\times\boldsymbol{B}']\,\mathrm{d}V, \tag{7.4.2}$$

in which μ_0 is the magnetic permeability of vacuum; the linearized Eulerian perturbation to the equilibrium magnetic field, $\boldsymbol{B}$, is $\boldsymbol{B}' = \operatorname{curl}(\boldsymbol{\xi}\times\boldsymbol{B})$, in the absence of magnetic diffusion. In eq. (7.4.1) the integrals I and K are defined in terms of the pressure and density distribution of the equilibrium state in the absence of $\boldsymbol{B}$. The quantities I_B and K_B are the linearized perturbations to I and K at constant $\boldsymbol{\xi}$ arising from the static perturbations to p, ρ and γ of the equilibrium state. The integral $\mathcal{M}$ represents the direct effect of the perturbed Lorentz force on the dynamics of the oscillations. Formally all the terms in eq. (7.4.1) arising from $\boldsymbol{B}$ are of the same order, namely $\bar{B}^2R^4/\mu_0GM^2$ times the corresponding unperturbed terms, where $\bar{B}$ is a characteristic value of $|\boldsymbol{B}|$. Thus the situation differs from that in section 7.3, where in general the perturbation is dominated by the direct effect of advection, the centrifugal force that distorts the equilibrium state in that case being of higher order in the perturbation.

In the way of the previous section the integral equation (7.4.1) can be linearized in the magnetic perturbation. The perturbation ω_B to the eigenfrequency is thus given by

$$\omega_B = \mathcal{M} + \tfrac{1}{2}(K_B - I_B\omega^2), \tag{7.4.3}$$

the integrals being evaluated with appropriate eigenfunctions of the corresponding nonmagnetic equilibrium configuration referred to an appropriately stretched independent variable, as was described in section 7.2. The appropriate zero-order eigenfunctions are each sums of eigenfunctions $\boldsymbol{\xi}_{lm}$ with the same order and degree, as in eq. (7.2.3).

The evaluation of ω_B is discussed for certain simple magnetic-field configurations by Goossens (1972) and Gough and Thompson (1990). I will not reproduce the details here, since the formulae are cumbersome. I remark simply that if the magnetic-field configuration is axisymmetric, about the axis of symmetry the modes are represented simply by eigenfunctions $\boldsymbol{\xi}_{nlm}$ of the form (5.1.1), as was the case in section 7.2. If, e.g., the equilibrium magnetic field is either azimuthal, of the form

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_{\mathrm{t}k}(\boldsymbol{r}) = \left[0,\ 0,\ \beta_\phi(r)\frac{\mathrm{d}}{\mathrm{d}\theta}P_k(\cos\theta)\right], \tag{7.4.4}$$

or poloidal, of the form

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_{\mathrm{p}k}(\boldsymbol{r}) = \left[k(k+1)\frac{\beta_{\mathrm{p}}(r)}{r^2}P_k(\cos\theta),\ \frac{1}{r}\frac{\mathrm{d}\beta_{\mathrm{p}}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}\theta}P_k(\cos\theta),\ 0\right], \tag{7.4.5}$$

it can be shown that the perturbed eigenfrequency is of the form

$$\Delta\omega_{nlm} = B_{mm} = \sum_{\lambda=0}^{2k} Q_{\lambda lm} I_\lambda^{\mathrm{mag}}, \tag{7.4.6}$$

where the quantities $I_\lambda^{\mathrm{mag}}(\boldsymbol{\xi}_{nlm}, \boldsymbol{\xi}_{nlm}^*)$ are integrals containing those components of I_B, K_B and $\mathcal{M}$ whose integrands contain products of eigenfunctions $\boldsymbol{\xi}_{nlm}$ and $\boldsymbol{\xi}_{nlm}^*$ (or their derivatives) multiplied by $P_\lambda(\cos\theta)$. The quantities B_{mm} introduced in eq. (7.4.6) represent the nonzero components of a diagonal matrix $B_{mm'}$, by analogy with the matrix $A_{mm'}$ defined by eq. (7.2.9). Notice that even if $\boldsymbol{B}$ is an odd function of $\cos\theta$, the Lorentz force, which is quadratic in $\boldsymbol{B}$, is even, and therefore gives a nonzero contribution to the frequency perturbation. It follows also that, since the Lorentz force is quadratic in $\boldsymbol{B}$ there can be no distinction between $\boldsymbol{B}$ and $-\boldsymbol{B}$. Therefore the system is invariant under reflections about the equator, which implies that there can be no distinction between eastward and westward propagating waves. Consequently the frequency perturbation must be independent of the sign of m, as is evident from eq. (7.4.6); it follows also from eq. (7.2.11) that the sum over m of the frequency perturbations to the modes with the same n and l is zero. More generally, one deduces from the symmetry of the Lorentz force that if $\boldsymbol{B}$ were not axisymmetric, the eigenfrequencies would remain degenerate in pairs, as is the case for the nonaxisymmetric scalar perturbations discussed in section 7.2. The eigenvalue equation is eq. (7.2.8) with $A_{mm'}$ replaced by $B_{mm'}$, except that now the matrix $\boldsymbol{B}$ is no longer diagonal.

It is useful to record the form of the perturbations for asymptotic modes. The terms K_B and I_B arise from the scalar perturbations to the equilibrium state produced by the equilibrium Lorentz force. They have already been discussed in section 7.2. There remains the term $\mathcal{M}$, arising directly from the perturbed Lorentz force. For an equilibrium toroidal magnetic field configuration of the form (7.4.4),

$$\mathcal{M} \sim S_{klm} \int_{r_1}^{R} \mathcal{K} \frac{v_A^2}{c^2}\, \mathrm{d}r \qquad (7.4.7)$$

for high-frequency p modes, where $v_A^2 = \beta_\phi^2/\mu_0\rho$ measures the square of the Alfvén speed in the equilibrium state and the kernel $\mathcal{K}$ is defined by eq. (7.1.18). The geometrical factor is defined by

$$S_{klm} = \left(l + \frac{1}{2}\right) \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} (1-\mu^2) \left(\frac{\mathrm{d}P_k}{\mathrm{d}\mu}\right)^2 [P_l^m(\mu)]^2\, \mathrm{d}\mu. \qquad (7.4.8)$$

An asymptotic approximation to this result for modes with $l \gg 2k$, together with an asymptotic formula for the frequency perturbation due to a poloidal field, is presented in section 8.7.4.

7.5. Adding perturbations

The preceding discussions have dealt separately with perturbations caused by different agents. What if there are several such perturbations together, as, e.g., in a rotating magnetic star?

The first remark I should make, which is very important, is that for the analysis to be valid there must exist a coordinate frame, $\mathcal{S}$, in which the basic (equilibrium) structure of the star is steady. Otherwise, time-dependent perturbation theory must be employed. In a rotating magnetic star, e.g., one would expect $\boldsymbol{B}$ to be advected

by the rotation velocity. However, one can imagine, e.g., that the region of the star containing the magnetic field is rotating rigidly, with angular velocity Ω_c say, so that if $\mathcal{S}$ rotates with angular velocity Ω_c the magnetic field could be steady in $\mathcal{S}$. Outside the field-containing region, the star might be rotating steadily with a nonuniform angular velocity relative to an inertial frame, and, provided that this angular velocity is about the same axis as Ω_c, the flow would also be steady in the frame $\mathcal{S}$.

The case of an axisymmetric equilibrium magnetic field $\boldsymbol{B}$ whose axis of symmetry coincides with the rotation axis, is straightforward. Then a possible choice of $\mathcal{S}$ is the inertial frame whose origin is the centre of the star. With respect to such a frame the components ξ_{nlm} of the fundamental representation (5.1.1) are separately eigenfunctions of both the corresponding rotating nonmagnetic configuration and the nonrotating magnetic configuration. It is evident that they are eigenfunctions of the rotating magnetic configuration too; therefore the total frequency perturbation, $\Delta\omega_{nlm}$, is obtained simply by adding the contributions (7.3.6) and (7.4.6).

If the magnetic symmetry axis does not coincide with the rotation axis, the situation is more interesting. Now it is essential to work in the rotating frame $\mathcal{S}$ in which $\boldsymbol{B}$ is steady. In this frame the frequency perturbations cannot simply be added, since in the forms presented in eqs. (7.3.6) and (7.4.6) they given with respect to different coordinate systems. If, e.g., one works in spherical polar coordinates, choosing the polar axis to coincide with the axis of rotation (which I now call the coordinate frame $\mathcal{S}$), the magnetic field is not axisymmetric about the coordinate axis, and the functions ξ_{nlm} are no longer eigenfunctions. To leading order in the perturbations the eigenfunctions are linear combinations of ξ_{nlm}, as in eq. (7.2.3).

To determine the coefficients c_m of the expansion (7.2.3) and the perturbed eigenfrequencies, it is first necessary to transform the matrix $B_{mm'}$ from the coordinate frame $\mathcal{S}'$, in which $\boldsymbol{B}$ is axisymmetric, to the frame $\mathcal{S}$. This is accomplished by expressing the function $\xi'_{nlm'}$, whose form is given by eq. (5.1.1) with respect to spherical polar coordinates (r, θ', ϕ') about the magnetic axis, in terms of the coordinates referring to the rotation axis:

$$\xi'_{nlm'} = \sum_m d^{(l)}_{mm'} \xi_{nlm}. \qquad (7.5.1)$$

The coefficients $d^{(l)}_{mm'}$ are given, e.g., by Edmonds (1957). They depend on the angle β between the two coordinate axes and the origins of ϕ and ϕ'. I choose the planes $\phi = 0$ and $\phi' = 0$ to coincide; then $d^{(l)}_{mm'}$ is real. Moreover, $\xi_{nlm} = \sum_{m'} d^{(l)}_{mm'} \xi'_{nlm'}$. The diagonal matrix $B'_{mm'}$ in eq. (7.4.6) (I have renamed it to $\boldsymbol{B}'$ to indicate that the values of the components refer to the coordinate frame $\mathcal{S}'$)

thus transforms to

$$B_{mm'} = \sum_{i,j} d^{(l)}_{mi} B'_{ij} d^{(l)}_{m'j} \tag{7.5.2}$$

in the coordinate frame $\mathcal{S}$. Of course, the matrix $B_{mm'}$ could alternatively be computed directly by expressing B with respect to the coordinates (r, θ, ϕ) in $\mathcal{S}$ and evaluating separately the elements from the products of the components of the expansion (7.2.3), but that would require more work. The eigenfunctions and perturbed eigenfrequencies are then determined, according to the arguments in section 7.2, by the eigenvalue equation analogous to eq. (7.2.8), namely

$$\sum_{m'} B_{mm'} c_{m'} + I^{-1}\mathcal{R} c_m - \Delta\omega^{kS}_{nl} c_m = 0, \tag{7.5.3}$$

where $I^{-1}\mathcal{R}$ is the right-hand side of the analogue of eq. (7.3.6) in the rotating frame $\mathcal{S}$. (The appropriate expression could be derived by repeating the analysis of section 7.2 in the rotating frame, which requires including the Coriolis force. However, it is simpler to note what the result must be by transforming the frequency perturbation (7.3.6) to the rotating frame: all the integrals with $\lambda \neq 0$ in the sum remain unchanged and the integral for $\lambda = 0$ is reduced by $I\Omega_c$.) Recall that the superscript k here labels the solution $(\Delta\omega^{kS}_{nl}, c^k_m)$, the superscript having previously been omitted from c_m for simplicity; it is not the degree of the Legendre polynomial used to define the simple magnetic-field configurations (7.4.4) and (7.4.5). The label $\mathcal{S}$ has been added to make clear that the frequency is measured in the rotating frame $\mathcal{S}$. Notice also that the method of analysis outlined in this section does not actually require the magnetic field to be of the simple forms (7.4.4) or (7.4.5), nor even to be axisymmetric; if it were not axisymmetric, the matrix $B'_{mm'}$ would no longer be diagonal, but eq. (7.5.3) would still be valid. Moreover, one can simply add to eq. (7.5.3) terms arising from other perturbations, such as the term $A_{mm'}c_{m'}$ of eq. (7.2.8), resulting from additional scalar perturbations to the equilibrium state that do not arise from magnetic effects.

Finally, let us recall that the frequency perturbations $\Delta\omega^{kS}_{nl}$ are referred to the rotating frame $\mathcal{S}$. When they are referred to the inertial frame, the multiplicity of the degeneracy splitting is increased further. The reason is clear. In the frame $\mathcal{S}$ a magnetic field or a scalar asphericity splits the degeneracy into $l + 1$ different eigenfrequencies, the eigenfunctions associated with each being in general the linear combination (7.2.3) of all $2l + 1$ degenerate eigenfunctions of degree l of the unperturbed spherically symmetric equilibrium state. The transformation to the inertial frame of the frequency associated with each component ξ_{nlm} of the combination is

$$\Delta\omega^k_{nlm} = \Delta\omega^{kS}_{nl} + m\Omega_c. \tag{7.5.4}$$

Thus the frequencies $\omega_{nl} + \Delta\omega^{kS}_{nl}$ of each mode in $\mathcal{S}$ are split into $2l + 1$ separate components, yielding a multiplicity of $(l + 1)(2l + 1)$ in the inertial frame.

7.6. Horizontal inhomogeneity

The predominant asphericities in real stars are often more complicated than the simple examples I have discussed in sections 7.2–7.4. In particular, a scalar perturbation in late-type stars is likely to be greatest in the convective envelopes, where it is associated with velocity fields of considerably greater complexity than the simple rotation discussed in section 7.3. If one ignores magnetic fields, and for the moment the inevitable time-dependence of the convective flow, one can proceed as in the preceding discussion by expanding the convective (and rotational) velocity and its associated scalar (e.g. c^2 and γ) perturbations in spherical harmonics, and computing the resulting degeneracy splitting of the normal modes of the corresponding unperturbed spherically symmetrical equilibrium state. The outcome is formally an ensemble of many closely spaced, discrete frequencies associated with each spherical harmonic component (l, m) of the oscillatory motion, some of which result from the frequency perturbations discussed above, the others being the frequencies of other modes whose perturbed eigenfunctions have spherical-harmonic components (l, m). In practice, these frequencies could never be measured individually (except, perhaps, those of the lowest-degree modes), since the convective flow is not steady on the observation time scale that would be necessary for resolving them. The difficulties are compounded by the fact that at any instant one can view only one side of a star, even, at present, in the case of the Sun, so one cannot unambiguously isolate the separate spherical-harmonic components. What can be measured, of course, is the shape and position of the broadened lines in the oscillation power spectrum to which the unresolved, discrete components contribute. Therefore, in principle, one could carry out the (time-dependent, if necessary) normal-mode perturbation analysis, and compute the outcome of the observations. For all but the lowest-degree modes, such a calculation is tedious, and in any case no doubt is not the most effective way of characterizing the oscillations. It is probably more prudent to concentrate on modes for which there is a considerable difference between the horizontal and temporal scales of variation of the mode and of the inhomogeneity of the basic state. (By "basic state" I mean the nonoscillating state, which, in the presence of time-dependent convection, is not strictly an equilibrium state.) Of course one can envisage situations where that is not possible, but I will not dwell on those here. Two situations naturally arise: that in which the spatial horizontal scale, λ, of the inhomogeneity is much less than that of the oscillation $(L^{-1}R)$ and that in which it is much greater. Both of these could be similarly subdivided according to temporal scales. Each case can then be analyzed by performing a scale separation, the evolution of the small-scale motion being expanded about the case where the large scale is infinite, its mean properties then being used to determine the evolution of the large-scale motion, as in the method of averages for determining adiabatic invariants of dynamical systems (e.g. Landau and Lifshitz (1960)).

In the case where the spatial scale of the dominant convective motion, e.g., is much less than that of the oscillations, one can average the convection over an intermediate scale, yielding the familiar Reynolds-stress and convective heat-flux terms in the leading-order equations of motion describing the dynamical oscillations of the star. To close the system of equations it is necessary to have a theory of time-dependent convection; this is sadly lacking, although mixing-length formalisms used to compute static stellar models, have been generalized for the purpose.

In the other extreme the convection is on a scale much larger than the oscillations, and JWKB (Liouville–Green) theory can be used to describe the horizontal variation of the latter. Under these circumstances the characteristic time scale of the convection is much greater than the periods of oscillation, at least for p modes, so JWKB theory can also be used to develop the temporal evolution of the oscillatory motion. Of course, to complete the description of the entire motion the transport properties of the oscillations so calculated should then be entered into the equations of motion for the convection. However, since to leading order for the oscillation equations no understanding of the dynamics of the convection is required, the convective perturbations to the oscillations can be computed in terms of an arbitrarily specified convective velocity and, say, pressure and density perturbations. (Care must be taken when interpreting the results, however, because only those frequency perturbations that arise from possible convective flows, which do satisfy the equations of motion, are actually realizable.) If these perturbations could be observed for the Sun, the theoretical relations might be used to invert the data to determine the large-scale convective flow, and thus, hopefully, further our understanding of the complicated dynamical processes that take place in the outer layers of stars.

What JWKB theory provides is a description of the ensemble of modes that one would observe with limited resolution. Thus, to leading order it represents the superposition of distorted eigenfunctions, each with its associated multiplicity of frequencies. It therefore provides directly and simply the tool one requires for inverting the data, through an average dispersion relation which relates the apparent frequency, $\bar{\omega}(\boldsymbol{r}, t)$, of a group of waves to their mean local horizontal wave number, $\bar{\boldsymbol{k}}(\boldsymbol{r})$. (Since $k := LR^{-1} \gg \lambda^{-1} \gtrsim R^{-1}$, I am restricted to high-degree modes, for which I can make an appropriate plane-parallel (sinusoidal) approximation to the spherical harmonics.) This dispersion relation contains two kinds of terms. One is that associated with the horizontal component, $\boldsymbol{U}$, of the convective velocity; it is simply an advection by an appropriately weighted average, $\bar{\boldsymbol{U}}$, of

$\boldsymbol{U}$, the weight depending on the mode of oscillation, and produces an equal and opposite frequency perturbation $\bar{\boldsymbol{k}} \cdot \bar{\boldsymbol{U}}$ to otherwise identical waves travelling in opposite directions. The other results from all other aspects of the structure of the star – the (horizontal) local variation of p, ρ, γ and the vertical component of the convective velocity – whose influence on the dispersion relation do not depend on the direction of $\bar{\boldsymbol{k}}$. Thus, from the symmetry properties of the dispersion relation one can immediately disentangle the effect of $\boldsymbol{U}$ from the other perturbations.

It is a straightforward matter to write down the formulae for the perturbations in this limit. I will not do this in all its generality. I simply want to demonstrate the equivalence of this approach and the normal-mode perturbation expansion. To this end I will oversimplify unrealistically, but without losing the essential physics I wish to illustrate, by considering the Lamb wave (the horizontally propagating acoustic wave mentioned in section 5.7.2 and appendix VI) propagating in a static plane-parallel atmosphere with constant γ, in which the sound speed, c, is independent of height.

To simplify matters still further, I will isolate just one Fourier component of the sound-speed variation, aligning the x-axis of a local Cartesian coordinate system with the direction of variation. Thus,

$$c^2 = c_0^2(1 - \varepsilon \sin \kappa x), \qquad (7.6.1)$$

where c_0, ε and $\kappa = \lambda^{-1}$ are constants. The basic state is assumed to be one with no motion, and must therefore satisfy hydrostatic support. In particular, $p = \gamma^{-1}\rho c^2$ is independent of x. Under these circumstances the linearized equations of motion of section 5 for a wave (mode) with frequency ω reduce to

$$\rho \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \delta p}{\partial x}\right) + \frac{\partial^2 \delta p}{\partial y^2} + \frac{\omega^2}{c^2}\delta p = 0 \qquad (7.6.2)$$

for the Lagrangian pressure fluctuation δp, and can be reduced to standard form (cf. sections 4.8 and 5.4) with the substitution $\delta p = \rho^{1/2}\Psi$, i.e.

$$\nabla_{\mathrm{h}}^2 \Psi + K^2 \Psi = 0, \qquad (7.6.3)$$

where ∇_{h}^2 is the horizontal Laplacian operator and

$$K^2 = \frac{\omega^2 - \omega_{\mathrm{l}}^2}{c^2}, \qquad (7.6.4)$$

$$\omega_{\mathrm{l}}^2 = \frac{c^2}{4\lambda^2}(1 - 2\lambda'), \qquad (7.6.5)$$

and the prime denotes differentiation with respect to x. Here λ is the horizontal density-scale length of the equilibrium state, defined with the usual sign convention: $\lambda := (-\partial \ln \rho/\partial x)^{-1}$. It is interesting to note the structural similarity between eqs. (7.6.3)–(7.6.5) and the plane-parallel limit of eqs. (5.4.7)–(5.4.9), particularly for solutions independent of y. Notice also that for eqs. (7.6.3)–(7.6.5) to be valid it is required that the equilibrium sound speed is a function of x alone, but it does not require the isolation of the single Fourier component (7.6.1).

If the expression (7.6.1) is introduced into eqs. (7.6.3)–(7.6.5), bearing in mind that λ is minus the scale length of c^2, and the outcome expanded to first order in ε, which now I presume to be small, there results

$$\nabla_{\rm h}^2\Psi + k^2\Psi \simeq -\varepsilon A\Psi \sin\kappa x, \tag{7.6.6}$$

where

$$k = \omega/c_0 \tag{7.6.7}$$

and

$$A = 2k^2 + \tfrac{1}{2}\kappa^2. \tag{7.6.8}$$

The solution to eq. (7.6.6) can then be obtained by expanding about a solution of the unperturbed state ($\varepsilon = 0$), which I take to be a plane Lamb wave, yielding

$$\begin{aligned}\Psi &= \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) + \frac{2\varepsilon A k_x}{\kappa(4k_x^2 - \kappa^2)}\\ &\quad\times\left[\cos\kappa x \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) - \frac{\kappa}{2k_x}\sin\kappa x\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right] &(7.6.9)\\ &= \mathcal{A}\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t - \delta), &(7.6.10)\end{aligned}$$

where $\boldsymbol{k} = (k_x, k_y)$ is the (horizontal) wave number, whose magnitude is k,

$$\begin{aligned}\mathcal{A} &= \left[\left(1 - \frac{\varepsilon A}{4k_x^2 - \kappa^2}\sin\kappa x\right)^2 + \frac{4\varepsilon^2 A^2 k_x^2}{\kappa^2(4k_x^2 - \kappa^2)}\cos^2\kappa x\right]^{1/2}\\ &\simeq 1 - \frac{\varepsilon A}{4k_x^2 - \kappa^2}\sin\kappa x, &(7.6.11)\end{aligned}$$

and

$$\begin{aligned}\delta &= \tan^{-1}\left[\frac{2\varepsilon A k_x}{\kappa(4k_x^2 - \kappa^2)}\cos\kappa x\left(1 - \frac{\varepsilon A}{4k_x^2 - \kappa^2}\sin\kappa x\right)^{-1}\right]\\ &\simeq \frac{\varepsilon A\cos\kappa x}{2\kappa k_x(1 - \kappa^2/4k_x^2)}. &(7.6.12)\end{aligned}$$

Alternatively, one can solve eq. (7.6.6) in the JWKB approximation by setting

$$\Psi_\pm = \mathcal{A}(x)\exp(\pm \mathrm{i}k_x\psi(x) \pm \mathrm{i}k_y y - \mathrm{i}\omega t), \tag{7.6.13}$$

with k large. The calculation is much simpler than the preceding method. The two leading equations are

$$\psi'^2 = 1 + \frac{\varepsilon A}{k_x^2}\sin\kappa x \tag{7.6.14}$$

from which

$$\psi \simeq x - \frac{\varepsilon A}{2\kappa k_x^2}\cos\kappa x, \tag{7.6.15}$$

and

$$\mathcal{A} = (\psi')^{-1/2}. \tag{7.6.16}$$

To first order in ε the combination $\frac{1}{2}(\Psi_+ + \Psi_-)$ is identical to the expressions (7.6.10), with $\mathcal{A}$ and δ given by eqs. (7.6.11) and (7.6.12) in the limit $\kappa/k_x \to 0$, which establishes the equivalence of the two methods of analysis, at least for this simple example. Both describe a motion which locally resembles a wave with frequency $\bar{\omega} = \omega$ (I have been able to use a representation with a well-defined frequency because my basic state is in static equilibrium) and a wave number whose x component is $\bar{k}_x(x) = k_x + (\varepsilon A/2k_x)\sin\kappa x$. Notice that what emerges as the spatial variation of the dispersion relation from the JWKB approximation arises from the spatial distortion to the eigenfunction in the perturbation expansion. This is the case whatever the nature of the perturbation. Thus it is evident, e.g., that a local asymptotic analysis of travelling waves in a rotating (or magnetic) star could lead to the detection of a north–south asymmetry in the angular velocity (or magnetic field). In a perturbation analysis of normal modes, one must investigate the first-order corrections to the zero-order eigenfunctions, which entails taking the analysis further than I have outlined in sections 7.2–7.4.

Strictly speaking the two methods I have discussed, are not equivalent, since the conditions for their validity are different. The perturbation expansion (7.6.9) requires that $\varepsilon A/(4k_x^2 - \kappa^2)$ and $\varepsilon A k_x/\kappa(4k_x^2 - \kappa^2)$ are small, with no further restriction on κ/k_x, whereas the JWKB approximation requires κ/k_x be small with no restriction on ε. The two analyses are therefore complementary. In the next section I develop the idea of representing the eigenfunctions locally as waves in the JWKB approximation further.

8. Asymptotic representation by locally plane waves

The Liouville–Green expansion, discussed in sections 4.8 and 5.8, makes a wave-like approximation to the radial variation of the eigenfunctions. Eigenfunctions are represented as standing waves formed by constructive interference of inward and outward propagating waves that are reflected at the turning points r_1 and r_2. Strictly speaking, the resulting approximations should be valid only when the order n of the mode is large, so that there are many wavelengths in any scale height of the equilibrium state between r_1 and r_2. In practice, however, the formulae often provide quite a good approximation even when n is only moderate. This property is well known from the JWKB approximation to other wave equations, such as the Schrödinger equation.

It is also possible to make a similar wave-like decomposition in three dimensions, by performing a Liouville–Green expansion on the basic equations of motion without any separation of variables of the type (5.1.1) and (5.1.2). The conditions for constructive interference, also known as resonance, are naturally rather more complicated, but the method has the advantage that it can be applied directly to nonspherical stars, when the eigenfunctions are not separable. In these lectures, however, I will apply it only to spherically symmetrical stars and to stars that are perturbed from the spherical state by only a small amount, reducing the formulae to the results obtained in sections 5 and 7. Except in some highly idealized circumstances, the governing equations for stars which deviate substantially from spherical symmetry, would need to be solved numerically; and this has never been carried out. The method of imposing resonance conditions played an important role in the early development of quantum theory, and has since become known as Einstein–Brillouin–Keller (EBK) or semi-classical quantization, for reasons that soon will be made clear. It is currently used particularly by theoretical chemists for solving the Schrödinger equation for complicated molecules.

8.1. The adiabatic wave equation in standard form

I first seek a single equation that describes the motion for a scalar, dependent variable Ψ, analogous to eq. (5.4.7) but in three dimensions. I will assume a static "equilibrium" background state in which there is no motion and no magnetic field, so that the linearized perturbation equations governing the oscillations are given by eqs. (3.1), (3.6), (3.10) and (3.1.4). Bearing in mind that I will be dealing with short-wavelength asymptotics, I will, for simplicity of presentation, ignore the local variation of the gravitational acceleration, g, and make the Cowling approximation, $\Phi' = 0$. Thus I will be analyzing the oscillations in terms of locally plane waves on a locally plane-parallel background. Note that I refer to "background" rather than "equilibrium" state, since, as before, I assume only hydrostatic equilibrium, to ensure that the only motion on the dynamical time scale is the oscillatory motion under study; I permit the possibility of thermal disequilibrium, e.g., causing the background state to evolve on a time scale much longer than the characteristic period of the waves. Thus in sections 8.2 and 8.3 I perform the analysis

considering the coefficients of the wave equation to be slowly varying functions of time. However, for simplicity I will regard the background state to be static when deriving the simple wave equation (8.1.8), and therefore also in the explicit application of EBK quantization to stars.

The derivation of the wave equation follows the usual principles: eliminate $\boldsymbol{u}$ and ρ' from the equations of motion, leaving an equation for a single scalar variable, which here I take to be $\chi := \operatorname{div} \boldsymbol{\xi} = -\gamma^{-1} p^{-1}\, \delta p$. The details of the analysis follow closely the derivation by Lamb (1932) of the wave equation for a perfect gas. First one eliminates p' and ρ' from the momentum equation (3.6), using eqs. (3.10) and (3.11):

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \rho\, \nabla(c^2\chi + \boldsymbol{g}\cdot\boldsymbol{\xi}) + (c^2\chi + \boldsymbol{g}\cdot\boldsymbol{\xi})\nabla\rho - \boldsymbol{g}\rho\chi - \boldsymbol{g}\boldsymbol{\xi}\cdot\nabla\rho. \qquad (8.1.1)$$

As usual, I have omitted the subscript zero from quantities pertaining to the background state, and I have rewritten $\boldsymbol{u}$ in terms of $\boldsymbol{\xi}$ using eq. (3.1). I have also used the equation of hydrostatic support, $\nabla p = \boldsymbol{g}\rho$, constraining the stratification of the background state.

From the hydrostatic equation (and its curl) follows that $\boldsymbol{g}$, ∇p and $\nabla\rho$ are parallel. Therefore I can simplify eq. (8.1.1) by writing

$$\boldsymbol{g} = -g\boldsymbol{n}, \qquad \nabla p = -g\rho\boldsymbol{n}, \qquad \nabla\rho = -H^{-1}\rho\boldsymbol{n}, \qquad (8.1.2)$$

where $\boldsymbol{n}$ is an upward directed unit vector and H is the density-scale height, yielding

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \nabla(c^2\chi - g\boldsymbol{n}\cdot\boldsymbol{\xi}) - \Gamma\chi\boldsymbol{n}, \qquad (8.1.3)$$

where

$$\Gamma = H^{-1}c^2 - g = g^{-1}c^2N^2, \qquad (8.1.4)$$

with N the buoyancy frequency defined by eq. (5.1.6).

The vector equation (8.1.3) is a relation for the three components of the displacement $\boldsymbol{\xi}$. The aim now is to eliminate two of the dependent variables, leaving an equation essentially for the third. I do not wish to end up with an equation for the component of a vector, however, because that is coordinate dependent. Instead I seek an equation for the scalar χ. To this end one first takes the divergence of eq. (8.1.3):

$$\frac{\partial^2 \chi}{\partial t^2} = \nabla^2(c^2\chi - g\boldsymbol{n}\cdot\boldsymbol{\xi}) - \boldsymbol{n}\cdot\nabla(\Gamma\chi). \qquad (8.1.5)$$

This equation depends on the vertical component $\boldsymbol{n}\cdot\boldsymbol{\xi}$ of $\boldsymbol{\xi}$, an expression for which can be obtained in terms of χ by taking the double curl of eq. (8.1.3), thereby eliminating the gradient term on the right-hand side:

$$\boldsymbol{n}\cdot\frac{\partial^2}{\partial t^2}\,\mathrm{curl}(\mathrm{curl}\,\boldsymbol{\xi}) = \frac{\partial^2}{\partial t^2}(\boldsymbol{n}\cdot\nabla\chi - \nabla^2\boldsymbol{n}\cdot\boldsymbol{\xi}) = -g\nabla_{\mathrm{h}}^2(\Gamma\chi), \tag{8.1.6}$$

where ∇_{h}^2 is the horizontal Laplacian operator. Thus, if g is independent of t, eqs. (8.1.5) and (8.1.6) combine to yield a single equation for χ alone:

$$\frac{\partial^4\chi}{\partial t^4} - \frac{\partial^2}{\partial t^2}[\nabla^2(c^2\chi) - \boldsymbol{n}\cdot\nabla(H^{-1}c^2\chi)] - N^2\nabla_{\mathrm{h}}^2(c^2\chi) = 0. \tag{8.1.7}$$

This is the required generalization of Lamb's equation. It is not in the required form for a Liouville–Green expansion, however, since it implicitly contains odd derivatives of the dependent variable. These can be removed by means of the substitution $c^2\chi = \rho^{-1/2}\Psi$, yielding the desired equation:

$$c^{-2}\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathrm{c}}^2\right)\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2}{\partial t^2}\nabla^2\Psi - N^2\nabla_{\mathrm{h}}^2\Psi = 0, \tag{8.1.8}$$

where here

$$\omega_{\mathrm{c}} = \frac{c}{2H}(1 - 2\boldsymbol{n}\cdot\nabla H)^{1/2} \tag{8.1.9}$$

is the planar value of the critical acoustic frequency in section 5, defined by eq. (5.4.9). Aside from a constant factor, the relationship between the dependent variable Ψ and δp, i.e. $\delta p = -\rho^{1/2}\Psi$, is also the plane-parallel limit of the more accurate expression (5.4.6) of section 5 that was derived for a spherically symmetrical background state.

Equation (8.1.8) is the master equation that should be used for quantization. However, to keep the presentation simple, I will specialize to acoustic oscillations and work with the simpler equation obtained by neglecting N^2:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathrm{c}}^2\right)\Psi - c^2\nabla^2\Psi = 0. \tag{8.1.10}$$

This is essentially the classical wave equation, modified by the acoustical cutoff term ω_{c}^2, although, of course, ω_{c}^2 and c^2 are considered to be slowly varying functions of position.

8.2. The three-dimensional Liouville–Green expansion

As in the one-dimensional case, it is necessary for the validity of the approximation that the scale of variation of the wave is much smaller than that of the background

state. Let the ratio be characterized by Λ^{-1}, which is considered to be small. The principle is to substitute into eq. (8.1.10) a wave-like function of the form

$$\Psi = A\,\mathrm{e}^{\mathrm{i}\Lambda\Phi}, \tag{8.2.1}$$

where the amplitude $A(\boldsymbol{r},t)$ and the phase function $\Phi(\boldsymbol{r},t)$ are presumed to vary on a length scale comparable with the scale of variation of the background state, and successively equate to zero the coefficients of descending powers of Λ. Acknowledging the possibility that ω_{c} might be $O(\Lambda)$, as indeed we know it must be in the surface layers of the star, the two leading equations are:

$$\dot\Phi^2 - (\omega_{\mathrm{c}}/\Lambda)^2 - c^2\nabla\Phi\cdot\nabla\Phi = 0, \tag{8.2.2}$$

$$\frac{\partial}{\partial t}(A^2\dot\Phi) - c^2\,\mathrm{div}(A^2\nabla\Phi) = 0, \tag{8.2.3}$$

where the dot denotes the partial (Eulerian) derivative with respect to time. This can be regarded as the three-dimensional JWKB approximation.

8.3. The eikonal equation

It is convenient to define the quantities

$$\omega(\boldsymbol{r},t) := -\Lambda\dot\Phi, \qquad \boldsymbol{k}(\boldsymbol{r},t) := \Lambda\,\nabla\Phi, \tag{8.3.1}$$

which can be regarded as a local frequency and a local wave number. In terms of these quantities eq. (8.2.2) can formally be solved for ω in terms of $\boldsymbol{k}$ and the background state, yielding

$$\omega = W(\boldsymbol{k},\boldsymbol{r},t) = (\omega_{\mathrm{c}}^2 + c^2k^2)^{1/2}. \tag{8.3.2}$$

This is called the *dispersion relation.*

One can now proceed by the route of geometrical optics (e.g. Whitham (1974)). It follows immediately from the structure of eq. (8.3.1) that

$$\dot{\boldsymbol{k}} + \nabla W = 0, \tag{8.3.3}$$

irrespective of the explicit expression in eq. (8.3.2) for W, where the operator ∇ denotes the gradient taking into account the spatial variation of $\boldsymbol{k}$. Since $\boldsymbol{k}$ is the gradient of a scalar, and consequently curl $\boldsymbol{k} = 0$, with respect to Cartesian coordinates, x_i, after expanding the gradient in terms of partial derivatives with respect to x_i and k_j at constant k_j and x_i, respectively, where k_i is the ith component of $\boldsymbol{k}$, eq. (8.3.3) can formally be rewritten as

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} := \dot{\boldsymbol{k}} + \boldsymbol{v}\cdot\nabla\boldsymbol{k} = -\nabla W. \tag{8.3.4}$$

Here ∇ is the usual gradient operator; thus, on the right-hand side of the equation, it operates on W at constant $\boldsymbol{k}$. The quantity $\boldsymbol{v}$ has Cartesian components $v_i = \partial W/\partial k_i$; it is known as the *group velocity* of the wave. I should perhaps introduce at this point also the commonly used term *phase velocity*, whose Cartesian components are W/k_i. Unlike the group velocity, it is not a vector; it is the more useful concept of *phase slowness*, k_i/ω, that is a vector.

The components x_i of the position vector $\boldsymbol{x}(t)$ of a point moving with group velocity $\boldsymbol{v}$ satisfies

$$v_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial W}{\partial k_i}, \tag{8.3.5}$$

where, according to eq. (8.3.4),

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\partial W}{\partial x_i}. \tag{8.3.6}$$

Equations (8.3.5) and (8.3.6) are Hamilton's equations for the Hamiltonian W expressed in terms of the canonical variables x_i, k_i. Therefore the theoretical machinery of classical mechanics is available for describing the properties of the waves. In particular, in section 8.7, I will utilize the principle of least action to simplify the quantization conditions for a nearly spherical star. Equation (8.3.6) describes the evolution of the wave number, $\boldsymbol{k}$, at a point that moves along a *ray path*, which is determined by eq. (8.3.5). In geometric optics it is called the eikonal equation.

One can similarly obtain an equation for the evolution of the frequency, ω. Differentiating the dispersion relation (8.3.2) with respect to time along a ray yields

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial W}{\partial k_i}\frac{\mathrm{d}k_i}{\mathrm{d}t}. \tag{8.3.7}$$

In view of Hamilton's equations (8.3.5) and (8.3.6), the second and third terms on the right-hand side cancel, yielding

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial W}{\partial t}. \tag{8.3.8}$$

It follows immediately that if the function W is not explicitly dependent on time, then ω is constant along a ray. This statement is the analogue of the conservation of energy for a system moving in a time-independent potential in either classical or quantum mechanics. In all the applications I will discuss, it will be assumed that this condition holds, which permits the separation of variables

$$\Psi(\boldsymbol{r}, t) = \Psi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{8.3.9}$$

in terms of a well-defined frequency ω.

A similar deduction can be made from eq. (8.3.6): if W is not explicitly dependent on a particular Cartesian coordinate x_j, say, then the component k_j is constant along a ray. This is the asymptotic basis for the harmonic separation $\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{h}\cdot\boldsymbol{r}}$ for waves in, e.g., a plane-parallel atmosphere, where $\boldsymbol{k}_\mathrm{h}$ is a constant wave-number vector perpendicular to the direction of variation of the background state. Similar results hold for cylindrically symmetric and spherically symmetric background states. The transformation to spherical polar coordinates (r, θ, ϕ) is carried out in appendix IX; there is shown that if the background state is independent of ϕ, then

$$r\sin\theta k_\phi = M, \tag{8.3.10}$$

where M is a constant, and if the background state is spherically symmetric, then in addition to condition (8.3.10),

$$r^2k_\mathrm{h}^2 := r^2(k_\theta^2 + k_\phi^2) = L^2, \tag{8.3.11}$$

where L is a constant. It will be shown in section 8.5 that the quantization conditions in a sphere constrain the constants M and L to be simply related to non-negative integers, m and l say: $M = m$, which can be identified with the azimuthal order of the mode, and L is a representation of the quantity (5.1.3) with the same name that was introduced in eq. (5.1.1).

8.4. EBK quantization

The asymptotic representation of the solution of the wave equation (8.1.10) is obtained in terms of locally plane waves by the methods of semi-classical quantization developed by Einstein (1917), Brillouin (1926) and Keller (1958), which correct the earlier Bohr–Sommerfeld quantization conditions. An example of a ray path of a stellar wave, which is the trajectory of a point satisfying eqs. (8.3.5) and (8.3.6), is illustrated in fig. 7. The illustration is for a spherically symmetrical stellar model. From the usual arguments of central-orbit theory for a Hamiltonian system, the ray lies in a plane passing through the centre of the sphere and is confined between two radii, r_1 and r_2, defining two caustic spheres, which are envelopes of the rays. I will call the domain between r_1 and r_2, within which the ray lies, the region of propagation. In general, the path is not closed, and the ray comes arbitrarily close to every point in the space between r_1 and r_2. Then a single ray essentially fills the propagating region, and when all the appropriate resonance (quantization) conditions are satisfied, it interferes with itself to generate an eigenmode of oscillation. In principle, it is only under this condition that the procedure outlined below is valid, since otherwise a ray would occupy a lower-dimensional subspace of the region of propagation, and neighbouring rays would not necessarily constitute a regular solution of eq. (8.1.10) with a unique frequency, ω.

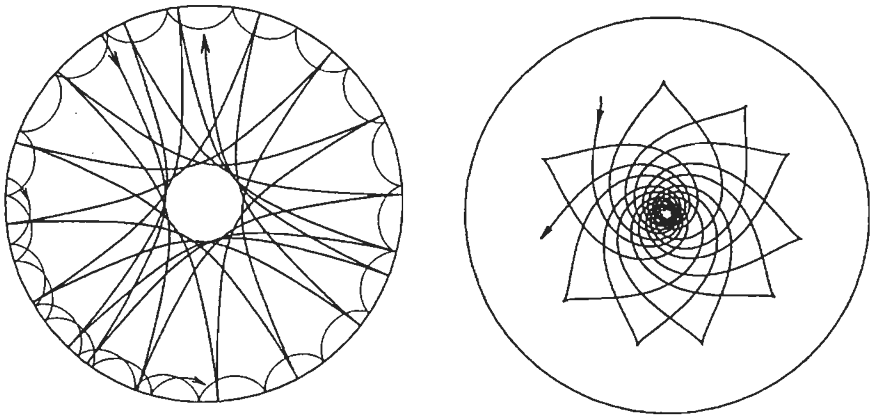


Fig. 7. Ray paths in the Sun. On the left are two acoustic rays; the more deeply penetrating ray is a constituent of $p_8(l = 2)$, with a cyclic frequency $\nu = \omega/2\pi = 1.38$ mHz, and the shallower ray is a constituent of $p_8(l = 100)$, with $\nu = 3.39$ mHz. On the right is a constituent of $g_{11}(l = 5)$, whose cyclic frequency is 0.19 mHz.

Since a ray in a spherically symmetrical system lies in a plane through the centre of symmetry, it cannot by itself cover the three-dimensional region of propagation. However, since all points on a sphere r = constant have the same values of ω_c and c, similar rays on different planes have essentially the same paths, and consequently it is possible to choose appropriate waves on neighbouring planes that match smoothly onto one another. Einstein (1917) regarded this case as simply the spherically symmetrical limit of a more general aspherical situation in which the ray paths genuinely fill the region.

Two classes of space-filling rays were contemplated by Einstein: (a) those for which, as the ray passes through an element of volume, $\mathrm{d}V$, there is only a finite number of values of $\boldsymbol{k}$, and (b) those for which there is an infinite number of values. Rays of the second class produce what is now called quantum chaos, and will not be considered here. I will consider only situations in which the waves are of class (a), and I assume that the stars of interest fall into that category too. Brillouin (1926) and Keller (1958) also considered only situations in this class. The general asymptotic solution of eq. (8.1.10) can then be represented as a finite sum of waves:

$$\Psi = \sum_{j=1}^{J} \Psi_j = \sum_{j=1}^{J} A_j \, \mathrm{e}^{\mathrm{i}\Lambda\Phi_j}, \tag{8.4.1}$$

where the amplitudes A_j and phases Φ_j separately satisfy equations of the type (8.2.2) and (8.2.3) for the same frequency $\omega = -\Lambda\dot{\Phi}_j$. Of course, I appreciate

that strictly speaking this is not an accurate representation of stars, which in reality have (time-dependent) convection zones which scatter the waves, but here I will ignore such scattering and deal only with idealized stellar models.

To make the discussion concrete, I will first discuss the oscillations of a spherically symmetrical star, and then discuss small perturbations from it. Thus, I start by considering rays in a plane such as that illustrated in fig. 7. The ray can be divided into segments of two distinct types: those propagating inwards and those propagating outwards. Thus there are just two ray segments, each travelling anticlockwise about the centre of symmetry of the background state, and that pass through any given point of a ray plane within the region of propagation. Therefore one can construct the components of the solution in that plane by setting $J = 2$. Of course, there is another similar pair of waves travelling clockwise, which in general should also be included. However, to keep matters simple, even when I consider axisymmetric perturbations from spherical symmetry, I will restrict attention separately to clockwise and anticlockwise waves. (Strictly speaking, I should keep all four waves in the plane and demonstrate that there is no coupling. However, it is simpler to anticipate the result, and show that a solution composed of only two waves can be found: we know from the discussion in section 7 that the eigenfunctions can be approximated by the separable forms (5.1.1) and (5.1.2), the coordinate axis being the axis of symmetry. Each function has a well-defined azimuthal order m, the sign of which determines whether azimuthal propagation is purely clockwise or purely anticlockwise. In the more general nonaxisymmetric case, however, the eigenfunctions cannot be described so simply. They are approximated by the linear combination (7.2.3) of separable functions with different values of m, and in general both negative and positive values are contained in the sum.)

The inward and outward ray segments are joined at the caustic surfaces. Rather than consider $\boldsymbol{k}$ to be double-valued in the region of propagation, Einstein (1917) found it convenient to consider $\boldsymbol{k}$ to be single-valued in a more complicated domain $\mathcal{D}$, in which the intersection of the plane of fig. 7 and the region of propagation is represented, in the way of Riemann sheets in complex function theory, by two surfaces joined at the caustics. Thus, the ray is considered to propagate outwards on, say, the upper sheet, and inwards on the lower sheet. In the case of a single ray for a spherically symmetric star, propagating in a plane, the domain $\mathcal{D}$ has the topology of a torus in three dimensions, and the ray spirals on its surface. It is simplest to keep at first this physically two-dimensional situation in mind. Einstein called the domain $\mathcal{D}$ the rational coordinate space; Keller used the now common parlance: covering space. Because $\boldsymbol{k}$ is single-valued in $\mathcal{D}$, and because, according to eq. (8.3.1), $\boldsymbol{k}$ is the gradient of a scalar, lines that are everywhere orthogonal to $\boldsymbol{k}$ exist. They are the intersections of the plane of fig. 7 and surfaces of constant phase, and they also spiral the torus, but with opposite helicity. The

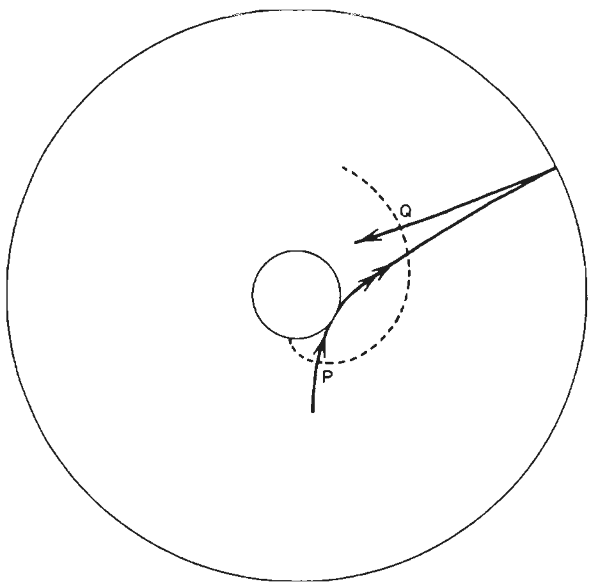


Fig. 8. The thick curve represents a segment of an acoustic ray, such as one of those illustrated in fig. 7. The dashed curve is the intersection of a surface of constant phase with the ray plane, and for the simplified dispersion relation (8.3.2) is orthogonal to the inward propagating rays, which are labelled with single arrow heads, intersecting them at P and Q. The circles are caustics.

condition that the solution Ψ to eq. (8.1.10) is a well-defined eigenstate is simply that $A\exp(i\Lambda\Phi)$ is single-valued. Consequently, a ray starting at P in fig. 8 and ending at Q on the same surface of constant phase must satisfy the quantization condition

$$\Lambda\,\delta\Phi = 2n'\pi + i\,\delta\ln A, \tag{8.4.2}$$

where n' is an integer and the symbol δ represents the difference of the values at Q and P.

Since the system is nondissipative, we may seek solutions for which A is independent of t. Then eq. (8.2.3) for the amplitude becomes

$$\mathrm{div}(A^2\boldsymbol{k}) = 2\boldsymbol{k}\cdot\nabla A + \Lambda A\,\nabla^2\Phi = 0, \tag{8.4.3}$$

whose solution is

$$A = A_0\exp\left(-\tfrac{1}{2}\Lambda\int k^{-1}\nabla^2\Phi\,\mathrm{d}s\right), \tag{8.4.4}$$

the integral being taken with respect to distance s along the ray path. Thus, since Φ is real, the phase of A does not vary along the ray, except possibly at the caustics, where $\nabla^2\Phi$ diverges.

The fundamental contribution that Keller (1958) made to the theory was to determine how the phase of A changes at a caustic surface. Roughly speaking, the conservation equation (8.4.3), or equivalently eq. (8.2.3) with $\partial/\partial t = 0$, implies that

$$A^2k\sigma = \text{constant} \tag{8.4.5}$$

along a tube of rays with cross section σ. At a simple caustic surface the rays cross on a line, and σ changes sign. Consequently A^2 changes sign (provided k is not zero on the caustic) and the phase of A is retarded by $\frac{1}{2}\pi$.

One might wonder why in this argument it is that the phase of A is retarded and not advanced by $\frac{1}{2}\pi$. To see this, one must examine the solution of the wave equation (8.1.10) in the vicinity of the caustic more closely. The result is well known in optics, and with a little thought one can readily convince oneself that the problem is a simple turning-point issue of the type I have already discussed in connection with the JWKB approximation; the retardation comes from demanding that the disturbance decays, rather than grows, exponentially away from the caustic in the evanescent region. One can easily convince oneself of the sign of the phase jump simply by drawing a sinusoid matched smoothly at the origin to an appropriately decaying exponential function.

Equation (8.4.2) can thus be rewritten:

$$\int_P^Q \boldsymbol{k}\cdot\mathrm{d}\boldsymbol{r} = 2\left(n' + \frac{m'}{4}\right)\pi, \tag{8.4.6}$$

where the path of integration is along a ray and where m' is the number of simple crossings of caustic surfaces encountered between P and Q. If a crossing were to occur at the confluence of two caustic surfaces, so that the cross section of the tube of rays is reduced to a point rather than a line (in other words, the dimensionality of the cross section is reduced by two, rather than one) the crossing would be counted twice.

Notice now that since $\boldsymbol{k}$ is perpendicular to the surfaces of constant phase, one can add to the integral on the left-hand side of eq. (8.4.6) an integral from Q to P along the line of constant phase in the covering space $\mathcal{D}$ without changing its value. Hence

$$\oint_C \boldsymbol{k}\cdot\mathrm{d}\boldsymbol{r} = 2\left(n' + \frac{m'}{4}\right)\pi, \tag{8.4.7}$$

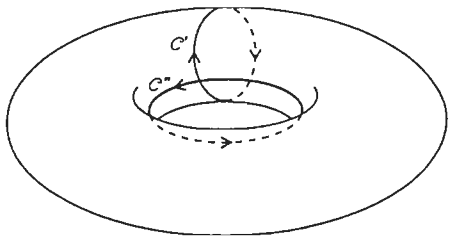


Fig. 9. Covering space for wave propagation in a plane with circular symmetry between two caustic circles, as in fig. 7, showing the contours C' and C''.

where C is a closed contour that winds around the torus $\mathcal{D}$, formed by a segment of a ray and a segment of a line of constant phase.

The condition (8.4.7) must be satisfied for all points P and Q in $\mathcal{D}$. That does not lead to an infinite number of independent quantization conditions, however. The number of independent quantization conditions is finite, and is equal to the number of degrees of freedom of the system. As Einstein (1917) pointed out, for a space-filling ray (i.e., a ray that fills the region between r_1 and r_2 in the plane of propagation) one can now continuously deform the contours C at will, provided, of course, they remain on the surface of the torus; it follows from eq. (8.3.1) that curl $\boldsymbol{k} = 0$, and therefore from Stokes' theorem that the values of the integrals are invariant. I should point out also that for such a generalization of the contour C the direction of the crossing of a caustic must be taken into account, and the sign of the contribution to m' assigned accordingly. If a contour can be contracted to a point, the integral is zero, m' is zero, and n' must be zero: the quantization condition provides no useful information. Otherwise, the contour can be deformed to a combination of integral numbers of basis curves of the covering space. Consequently, the integrals around all the basis curves provide a complete set of quantization conditions. These are the analogues of the correct expressions for the Bohr–Sommerfeld conditions in quantum theory.

To see more clearly how these conditions arise, let us first specialize to two dimensions, seeking the solution of eq. (8.1.10) in the area enclosed by a circle. The ray in fig. 7 then constitutes the complete solution. The entire covering space is the torus depicted in fig. 9, whose basis curves are C' and C''. Therefore there are two independent quantization conditions. The caustics are the curves concentric with C'' having the minimum and maximum radius in $\mathcal{D}$. The curve C' crosses each caustic once and C'' crosses neither. Hence, the two independent quantization conditions are given by

$$\oint_{C'} \boldsymbol{k}\cdot \mathrm{d}\boldsymbol{r} = 2\left(n' + \frac{1}{2}\right)\pi, \qquad (8.4.8)$$

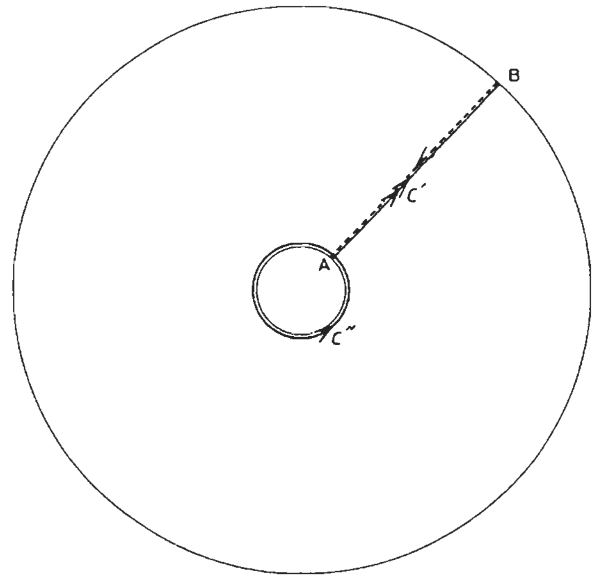


Fig. 10. Projection of fig. 9 onto the ray plane.

$$\oint_{C''} \boldsymbol{k}\cdot \mathrm{d}\boldsymbol{r} = 2l'\pi, \qquad (8.4.9)$$

for nonnegative integers n' and l'. The contours C' and C'' are shown projected onto the physical two-dimensional configuration space in fig. 10. One is purely in the radial direction and the other is a circle concentric with the caustics. Of course, C' and C'' could be deformed further, but usually the contours as drawn in the figure are the most convenient to adopt for evaluating the integrals in conditions (8.4.8) and (8.4.9).

The generalization to oscillations of a sphere is discussed by Keller and Rubinow (1960). Now a solution can be found which is composed of a superposition of plane rays obtained by rotating the plane of figs. 7, 8 and 10 about the axis of spherical polar coordinates (r, θ, ϕ). As will become apparent, to satisfy the quantization conditions, the angle $\frac{1}{2}\pi - \theta_0$ between the coordinate axis and the normal to the planes must take only specific quantal values. Moreover, the phases of rays on neighbouring planes must be appropriately related, so that the dashed line in fig. 8 maps out a unique surface within the region of propagation as the plane of the figure is rotated. Figure 11 illustrates the plane of fig. 10 imbedded in the sphere. By considering all points P between the caustics of the plane as it is rotated, it is clear that the waves occupy that region of space between the spherical surfaces

$r = r_1$ and $r = r_2$, forming the caustic spheres, that lies outside the cone with semi-angle θ_0 about the axis. Evidently, the cone is also a caustic surface. The covering space is the three-dimensional surface in the four-dimensional space defined by rotating the torus in fig. 9 about the coordinate axis: topologically, $\mathcal{D}$ is the Cartesian product of a torus and a circle. There are now three independent basis curves, which may be taken to be C' and C'' of figs. 9 and 10, and, say, the circle C''' within the region of propagation and arbitrarily close to the intersection of the cone $\theta = \theta_0$ with the inner caustic sphere $r = r_1$. The quantization condition (8.4.8) is unchanged, because the contour C' still intersects two caustics, which are now the spheres $r = r_1$ and $r = r_2$. Condition (8.4.9) is modified, however, because the contour C'' now intersects the caustic cone twice. C''' touches no caustic. Therefore the quantization conditions are

$$\oint_{C'} \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{r} = 2\left(n' + \frac{1}{2}\right)\pi, \quad n' = 0, 1, 2, \ldots, \tag{8.4.10}$$

$$\oint_{C''} \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{r} = 2\left(l + \frac{1}{2}\right)\pi, \quad l = 0, 1, 2, \ldots, \tag{8.4.11}$$

$$\oint_{C'''} \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{r} = 2m\pi, \quad m = 0, 1, 2, \ldots. \tag{8.4.12}$$

8.5. Evaluation of the quantization conditions

Evaluation of conditions (8.4.10)–(8.4.12) for a spherically symmetrical star is straightforward. In view of the restrictions (8.3.10) and (8.3.11) on the azimuthal component k_ϕ and the total horizontal component k_h of the wave number under these circumstances, conditions (8.4.11) and (8.4.12) become

$$2\pi r_1 k_h = 2\pi L = 2\pi(l + \tfrac{1}{2}) \tag{8.5.1}$$

and

$$2\pi r_1 \sin\theta_0 \, k_\phi = 2\pi M = 2\pi m. \tag{8.5.2}$$

Thus

$$L = l + \tfrac{1}{2} \tag{8.5.3}$$

and $M = m$. The radial component, k_r, required for evaluating the quantization condition (8.4.10) can now be obtained from k_h and the total wave number k,

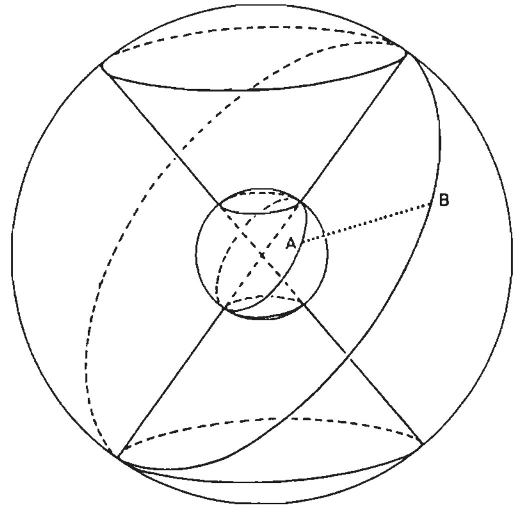


Fig. 11. Imbedding of fig. 10 in a sphere. The dotted line between A and B represents the contour C', which lies in the ray plane between the two spherical caustic surfaces which bound the region of propagation. Shown also is the caustic cone, which is the envelope of the plane of fig. 10 as it is rotated about the coordinate axis (which is not shown), and inside of which the waves are evanescent.

which is given in terms of the frequency, ω, and the structure of the star by the dispersion relation (8.3.2). Thus, condition (8.4.10) becomes

$$\int_{r_1}^{r_2} K \,\mathrm{d}r := \int_{r_1}^{r_2} \left[\frac{\omega^2 - \omega_c^2}{c^2} - \frac{L^2}{r^2}\right]^{1/2} \mathrm{d}r = \left(n - \frac{1}{2}\right)\pi, \tag{8.5.4}$$

where I have set $n = n' + 1$, so that counting starts from $n = 1$, in accord with the classification discussed in section 5.5. Equation (8.5.4) here provides a definition of K, which is simply the vertical component of the wave number, and is essentially the same as that given by eq. (5.4.8) with $\mathcal{N}^2$ neglected. This equation is essentially eq. (5.8.1), with $\mathcal{N}$ and ϵ ignored, which was derived from the JWKB approximation to the separable solution (5.1.1), (5.1.2), except that now the definition (5.1.3) of L has been replaced by the slightly different quantization condition (8.5.3). Had I worked from the dispersion relation obtained from eq. (8.1.8), rather than the more highly simplified relation (8.3.2) derived from eq. (8.1.10), I would have included a contribution from buoyancy, yielding eq. (5.8.1) with $\epsilon = 0$ and $\mathcal{N}$ replaced by its planar value N. (Of course, I could even have included the spherical geometry in deriving eq. (8.1.8), in which case ω_c and $\mathcal{N}$ would have been correctly represented.) Strictly speaking, ray theory is valid only when all components of k are large compared with the inverse scale of variation of the background state. Therefore one should expect eq. (8.5.4) to be valid only when $l \gg 1$, so the difference between eqs. (5.1.3) and (8.5.3) should not be significant. One might expect eq. (5.1.3) to be superior, since the analysis leading to eq. (5.8.1) placed no restriction on the value of l. It is pertinent to recall, however, that, as mentioned in section 5.8, in practice eq. (5.8.1) or eq. (8.5.4) often provides a closer approximation to the exact solution when eq. (8.5.3) is used for L. That can be true also when solving Schödinger's equation by asymptotic methods (e.g. Kemble (1937)).

Conditions (8.4.11) and (8.4.12) also determine the quantization of the angle θ_0 that the planes of figs. 7, 8 and 10 make with the coordinate axis. Since both the contours C'' and C''' lie essentially on the caustic $r = r_1$, they are always tangent to horizontal rays. Hence $k_\phi = k_h$ for the particular choice of C'' and C''' that I have made. Dividing condition (8.5.2) by eq. (8.5.1) thus yields

$$\theta_0 = \sin^{-1}\left(\frac{m}{l+1/2}\right). \tag{8.5.5}$$

Clearly, we can identify l and m with the degree and the azimuthal order of the separable solutions discussed in section 5. This will become even more evident in the next subsection, in which I construct the eigenfunctions. We note here simply that the waves avoid a cone about the coordinate axis, and that for sectoral modes with high degree, $m/(l+\frac{1}{2}) = l/(l+\frac{1}{2}) \simeq 1$ and the modes are localized close to the equatorial plane, lying between the latitudes $\pm\cos^{-1}[l/(l+\frac{1}{2})] \simeq \pm(l+\frac{1}{2})^{-1/2}$.

8.6. Construction of the eigenfunction

The eigenfunction Ψ is given by eq. (8.4.1); at any point P it is the sum of all the distinct waves Ψ_j that pass through P. As I deduced in the preceding two sections, for a spherically symmetrical star the ray paths lie on planes inclined at an angle θ_0 (given by eq. (8.5.5)) to the coordinate axis, on each of which there are two distinct ray segments through P. Furthermore, provided P lies outside the caustic cone, there are two planes with inclination θ_0 that pass through P; these are illustrated in fig. 13. To construct the eigenfunction, one must therefore compute the amplitudes A_j and the phases Φ_j of the four constituent waves Ψ_j. After ensuring that the separate values of A_j and Φ_j are appropriately related (which is accomplished by considering the matching conditions at the caustics; since each ray is space-filling in its plane, the four waves all lie essentially on different segments of the same ray, if we take Einstein's view that the extension of a ray from one orbital plane to another can be considered to be the same ray (strictly speaking, segments of a single ray that is not closed cannot all pass through P, but they come arbitrarily

close to P, and by continuity of Ψ their amplitudes and phases arbitrarily close to P determine the amplitudes and phases of ray segments that do pass through P), and their amplitude and phase relations are therefore well-determined), it is a straightforward matter to sum the waves Ψ_j to determine Ψ. I consider first the phases, then I compute the amplitudes and finally I construct Ψ.

8.6.1. The phases

The great circle of intersection of one of the planes of a ray path through a point P and a sphere with radius r (satisfying $r_1 < r < r_2$) concentric with the star, whose centre is O, is illustrated in fig. 12. Shown are Cartesian axes (x, y, z), with respect to which I define spherical polar coordinates (r, θ, ϕ). It is convenient to define two additional coordinate frames: (x', y', z') defined by rotating the original frame about the z-axis by an angle ϕ_0 such that the x'-axis coincides with the intersection of the ray plane with the equatorial plane of the original frame, and (x'', y'', z'') obtained by a further rotation about the x'-axis such that the $x''y''$-plane coincides with the ray plane; in the $x''y''$-plane I also introduce the polar angle ϕ'', satisfying $x'' = r\cos\phi''$. Thus, the ray lies in the plane $z'' = 0$.

Let the point P in the ray plane have coordinates (r, θ, ϕ) in the original frame; without loss of generality I assume the angle QOP to be less than $\frac{1}{2}\pi$, so the segment QP of the great circle does not touch the caustic cone. Let Q be the intersection of the ray plane and the equator of the sphere with positive x', and let Q' be its antipodal point. Also, let R be the intersection of the great circle passing through P and the poles in the (x, y, z) frame and the equator; it has coordinates $(r, \frac{1}{2}\pi, \phi)$. Finally, let S be the intersection of the x-axis with the sphere. I will use a notation for angles such that Φ_{QOP}, e.g., is the angle subtended at O by Q and P in the direction Q → P.

First I relate the phase Φ_j to the point T at $(r_1, \frac{1}{2}\pi, 0)$, the intersection of the x-axis with the inner caustic sphere, at which I presume the phase to be Φ_T. This point T is not indicated in fig. 12 since it is of no material importance. The phase Φ_j with respect to T is now given by

$$\Phi_j - \Phi_T = \int_T^P \boldsymbol{k}\cdot d\boldsymbol{r} = \int_T^S k_r\, dr + \int_S^P \boldsymbol{k}_h \cdot d\boldsymbol{r} =: \Phi_{rj} + \Phi_{hj}, \tag{8.6.1}$$

Φ_{rj} and Φ_{hj} being the integrals from T to S and from S to P, respectively. For the outward propagating wave, Ψ_1, the phase component Φ_{r1} is obtained by integrating directly from T to S:

$$\Phi_{r1} = \int_{r_1}^{r} K\, dr - \frac{\pi}{4}, \tag{8.6.2}$$

where $K = |k_r|$, in which, for symmetry, I have included half the phase retardation at the caustic sphere $r = r_1$. For the inward wave, Ψ_2, the path of the integral must

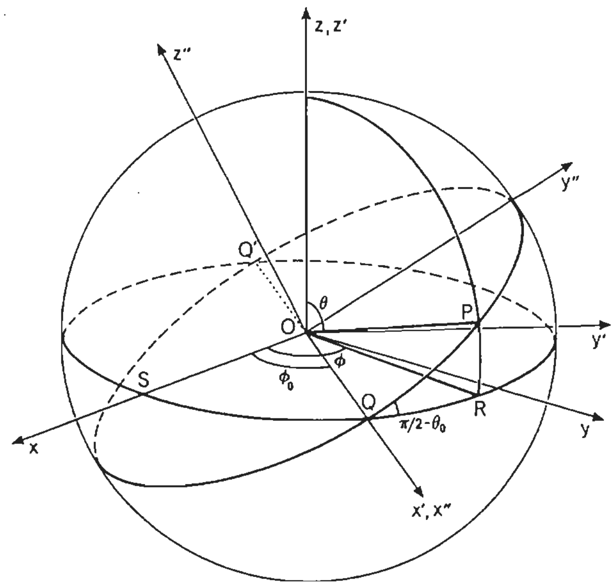


Fig. 12. Diagram showing the ray plane in a sphere of radius r and the coordinates used in the construction of the eigenfunction.

cross the outer caustic (once is sufficient) and join with S without encountering the inner caustic on the way. In view of the quantization condition (8.5.4), the phase component Φ_{r2}, modulo an integral multiple of 2π (which is immaterial for the purposes of evaluating the eigenfunction) is $\Phi_{r2} = -\Phi_{r1}$.

The horizontal integral is the same for both waves: $\Phi_{h1} = \Phi_{h2}$. It is given by

$$\Phi_{h1} = \phi_0 L \cos(\tfrac{1}{2}\pi - \theta_0) + L\Phi_{QOP}, \tag{8.6.3}$$

since, according to eq. (8.3.11), $r^{-1}L$ is the magnitude of the horizontal component of the wave number and $r^{-1}L\cos(\frac{1}{2}\pi - \theta_0)$ is evidently the equatorial component at $\theta = \frac{1}{2}\pi$. Indeed, one can rewrite the first term on the right-hand side of eq. (8.6.3) as $m\phi_0$, either by substituting the quantization conditions (8.5.2), (8.5.5) and (8.5.3) into that term or by writing it directly as $\int r\sin\theta\, k_\phi\, \mathrm{d}\phi$ and using eqs. (8.3.10) and (8.5.2). It is also convenient to rewrite the second term as $m\Phi_{QOR} - m\Phi_{QOR} + L\Phi_{QOP}$, combining the first of these terms with $m\phi_0$ to obtain $m\phi$. The angles Φ_{QOR} and Φ_{QOP} can then be obtained from the coordinate transformations

$$y' = r\sin\theta\sin(\phi - \phi_0) = y''\sin\theta_0 = r\sin\phi''\sin\theta_0, \tag{8.6.4}$$

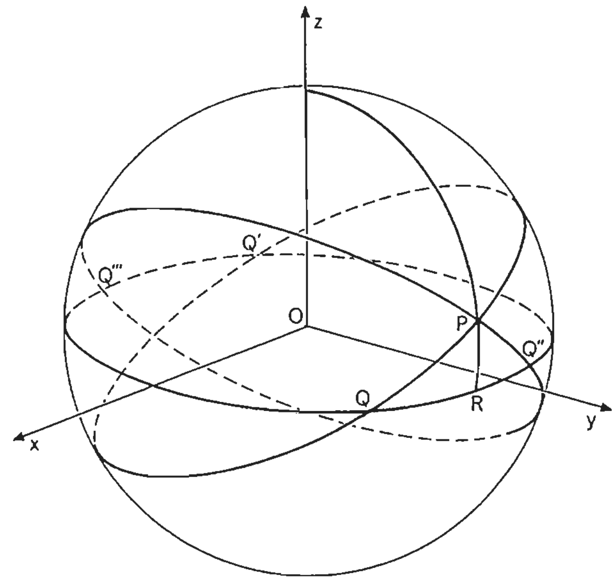


Fig. 13. Diagram showing the ray plane of fig. 12 and the other ray plane passing through the point P. The notation is the same as in fig. 12.

$$z' = r\cos\theta = r\mu = y''\cos\theta_0 =: r\sin\phi''\cos\theta_0, \tag{8.6.5}$$

in which I have used the fact that $z'' = 0$ in the ray plane. Dividing eq. (8.6.4) by eq. (8.6.5) yields

$$\Phi_{QOR} = \phi - \phi_0 = \sin^{-1}\left(\frac{\tan\theta_0}{\tan\theta}\right) = \sin^{-1}\left[\frac{m\mu}{L\mathcal{M}(1-\mu^2)^{1/2}}\right]; \tag{8.6.6}$$

one obtains Φ_{QOP} directly from eq. (8.6.5):

$$\Phi_{QOP} = \phi'' = \sin^{-1}\left(\frac{\cos\theta}{\cos\theta_0}\right) = \sin^{-1}\left(\frac{\mu}{\mathcal{M}}\right). \tag{8.6.7}$$

In these equations I have introduced the quantity $\mathcal{M}$, defined by

$$\mathcal{M} := \cos\theta_0 = (1 - m^2/L^2)^{1/2}, \tag{8.6.8}$$

the second relation having being obtained from eq. (8.5.5). These relations complete the determination of the phases Φ_1 and Φ_2 of the outward and inward waves propagating in the ray plane illustrated in fig. 12.

The phases Φ_3 and Φ_4 of the outward and inward waves Ψ_3 and Ψ_4, propagating on the other plane containing P and which also makes an angle θ_0 with the z-axis, illustrated in fig. 13, are constructed in a similar way. But before one can write down their values, it is necessary to establish the conditions for these waves to match correctly onto appropriate waves Ψ_1 and Ψ_2 at the caustics. Evidently, $\Phi_{r3} = \Phi_{r1}$ and $\Phi_{r4} = \Phi_{r2}$. Therefore it is only necessary to consider the matching of the horizontal contributions. To this end I first evaluate the phase component Φ_{h1} at Q′ from eqs. (8.6.3), (8.6.6) and (8.6.7) by setting $\theta = \frac{1}{2}\pi$; $\Phi_{QOR} = \Phi_{QOP} = \Phi_{QQ'} = \pi$, whence

$$\Phi_{h1}(Q') = L\pi - \tfrac{1}{2}\pi = l\pi; \tag{8.6.9}$$

notice the inclusion of the phase retardation of $\frac{1}{2}\pi$ resulting from passing the caustic cone at $\phi = \phi_0 + \frac{1}{2}\pi$. (One might also note, in passing, that the phase increment around the entire great circle through Q and P is $2l\pi$, which is an integral multiple of 2π, as it should be.) Thus, if Q″ is the intersection of the second ray plane with the equator such that the great-circle segment Q″P does not encounter the caustic cone, and Q‴ is its antipodal point, one obtains

$$\begin{aligned}\Phi_{h3} = \Phi_{h4} &= m\phi_0 - m\Phi_{QOQ'''} + l\pi + L\Phi_{Q''OP}\\ &= m\phi + m\Phi_{QOR} - L\Phi_{QOP} + (l-m)\pi\\ &= m\phi - (\Phi_{h1} - m\phi) + (l-m)\pi.\end{aligned} \tag{8.6.9double}$$

At this point, for tidiness, I now set $\Phi_T = -\frac{1}{2}(l-m)\pi$, which causes the constant contribution to be shared equally between all the phases. This yields

$$\Phi_{hj} + \Phi_T = m\phi \pm \left[L\sin^{-1}\left(\frac{\mu}{\mathcal{M}}\right) - m\sin^{-1}\left(\frac{m\mu}{L\mathcal{M}(1-\mu^2)^{1/2}}\right) - \frac{1}{2}(l-m)\pi\right], \tag{8.6.10}$$

where the principal value of $\sin^{-1}$ is implied; the plus sign refers to $j = 1$ and 2, the minus sign to $j = 3$ and 4.

8.6.2. The amplitudes

The amplitudes are obtained from the asymptotic conservation equation (8.4.5). It is necessary, therefore, to calculate the spatial variation of the cross section, σ, of a pencil of rays.

First consider waves on two neighbouring ray planes with the same inclination θ_0 whose intersections Q and Q′ (in the notation of fig. 12) with the equatorial plane are separated by a small difference $\delta\phi$ in the polar angle ϕ. The planes intersect on the y''-axis, and therefore the perpendicular distance between them is

$$x''\cos\theta_0\,\delta\phi = r\cos\phi''\cos\theta_0\,\delta\phi = r(\mathcal{M}^2 - \mu^2)^{1/2}\,\delta\phi, \tag{8.6.11}$$

the second of these relations having been obtained with the help of eq. (8.6.7). Now consider two neighbouring rays on one of these planes, one being generated from the other by a small rotation $\delta\phi''$ in the polar angle ϕ'' about the z''-axis. If ψ is the inclination of the ray from the vertical at radius r, then the perpendicular distance between the two rays is

$$r\cos\psi\,\delta\phi'' = rk^{-1}|k_r|\,\delta\phi'', \tag{8.6.12}$$

where, as before, $|k_r| = K$ is the magnitude of the vertical component of the wave number $\boldsymbol{k}$. The right-hand side of eq. (8.6.13) follows immediately after recalling that in this section I am ignoring the buoyancy frequency and that therefore the ray is parallel to $\boldsymbol{k}$; I am interested only in the magnitude of the distance between the rays, the sign having already been taken into account by Keller's phase retardation at the caustics.

For a given pencil of rays, $\delta\phi$ and $\delta\phi''$ are constant. Therefore, combining eqs. (8.4.5), (8.6.12) and (8.6.13) yields

$$A_j \propto r^{-1}K^{-1/2}(\mathcal{M}^2 - \mu^2)^{-1/4}. \tag{8.6.13}$$

Since eq. (8.4.5) also indicates that the amplitude is the same at equal small distances in the covering space at either side of a caustic, it follows that eq. (8.6.14) holds for all j, with the same constant of proportionality.

8.6.3. The eigenfunction

The phases and amplitudes computed in the previous two subsections give sufficient information to construct the eigenfunction Ψ. Substituting into eq. (8.4.1), after rewriting the phases (8.6.11) in terms of inverse cosines, yields

$$\Psi \sim \Psi_0 r^{-1}K^{-1/2}\cos\left(\int_{r_1}^{r} K\,\mathrm{d}r - \frac{\pi}{4}\right)\mathcal{P}_l^m(\mu)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{8.6.14}$$

where, assuming m to be non-negative,

$$\mathcal{P}_l^m(\mu) = (\mathcal{M}^2 - \mu^2)^{-1/4} \times \cos\left[L\cos^{-1}\left(\frac{\mu}{\mathcal{M}}\right) - m\cos^{-1}\left(\frac{m\mu}{L\mathcal{M}(1-\mu^2)^{1/2}}\right) - \frac{\pi}{4}\right] \tag{8.6.15}$$

and Ψ_0 is a constant; principal values of $\cos^{-1}$ are to be taken. In constructing these solutions, I considered only waves propagating around the coordinate axis in the direction of increasing ϕ. There is another eigenfunction composed of waves propagating in the direction of decreasing ϕ; it is easy to see from the preceding analysis that it is obtained simply by reversing the sign of m in the exponential in eq. (8.6.15).

It is interesting to note that the quantized eigenfunction (8.6.15) is separable in its coordinates ($r, \mu = \cos\theta, \phi$), even though separability was not explicitly assumed at the outset. (One can readily trace this property back to being the result of considering the superposition of waves only on planes inclined with the same angle θ_0 to the coordinate axis.) Recalling that here $\Psi = -\rho^{-1/2}\,\delta p$, one can see that it is similar to the separable solution for δp of the form (5.1.2), whose asymptotic radial component (for $\omega^2 \gg g/r$), obtained by the JWKB approximation, is given by eq. (5.8.5). The sole difference in the dependence on r arises from the fact that, for simplicity, I have used the ray theory only for high-frequency acoustic modes; to reduce eq. (5.8.5) to the component of expression (8.6.15) that depends only on r, it is necessary to ignore $\omega^{-2}\mathcal{N}^2$ compared with unity in eq. (5.4.8), ignore the geometrical terms in eq. (5.4.9) by omitting the last term and replacing $\mathcal{H}$ by H, and retaining only the dominant term $\omega^2 r/g$ in the expression (5.1.11) for the discriminant f. The dependence on θ differs only by the deviation of $\mathcal{P}_l^m$ from the associated Legendre function P_l^m. It is shown in appendix X that the JWKB approximation to P_l^m is asymptotically equivalent to $\mathcal{P}_l^m$ when l is large.

This construction provides an interpretation of the quantities L and l in geometrical terms. Roughly speaking, what I have done is represent the eigenfunction as a superposition of sectoral modes whose planes of symmetry all lie at an angle of approximately $\sin^{-1}(m/l)$ with the coordinate axis. As the plane is rotated about the coordinate axis, the phases of the constituent sectoral modes must be such as to produce constructive interference. Each sectoral mode can be regarded as the interference pattern of waves lying close to its equatorial plane: it is channelled by the spherical geometry of its container, which acts like a wave guide with half-width δr, intersecting the surface of the star in a band with half-width δR about a great circle. The wave has l wavelengths around the great circle, as is evident from the discussion leading to eq. (8.6.10). Thus the component of the horizontal wave number along the great circle is lR^{-1} and the component perpendicular to it is roughly $(\pi/2\delta)R^{-1}$. The square of the total horizontal wave number is therefore $(l^2 + \frac{1}{4}\pi^2\delta^{-2})R^{-2} \simeq L^2R^{-2}$, from which follows that $\delta \simeq \pi/2\sqrt{l} \simeq l^{-1/2}$, in agreement with the deduction from eq. (8.5.5). Indeed, this is just as one would expect from wave-interference arguments: it is straightforward to show that at any radius r the length of a circle of latitude at latitude δ, in a coordinate system orientated such that the ray path of the sectoral mode lies in the equatorial plane, differs from the length of the equator by half a horizontal wavelength of the mode.

8.7. Aspherical perturbation theory

Calculating the effect of small spherical perturbations is straightforward. The structure of the ray equations is not materially altered, and evidently the eigenfunctions remain separable, the angular dependence being unchanged. Thus the perturbations to the eigenfrequencies and eigenfunctions can be obtained simply by formally perturbing the formulae obtained in the previous two sections. On the other hand, when the perturbation to the basic state of the star is aspherical, the rays are additionally refracted; in particular, they are no longer planar, which increases the complexity of the geometry substantially. However, when the distortion is small, linearized perturbations to the oscillations can be expressed in terms of integrals along the unperturbed contours in section 8.4, and the problem is simplified enormously. As an introduction to how the analysis is carried out, I first derive a variational principle.

8.7.1. The principle of least action

Consider the function

$$\mathcal{S} = \int \mathcal{L}\,\mathrm{d}t, \tag{8.7.1}$$

where

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{k}, t) := \boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - W = \boldsymbol{k}\cdot\boldsymbol{v} - W, \tag{8.7.2}$$

W being given by eq. (8.3.2), and the integral in eq. (8.7.1) is taken along a ray path either between two fixed points in space or from one point of a surface, Σ, of constant phase to another. It does not matter whether or not the ray intersects Σ en route. Recall that under the approximation (8.3.2), $\boldsymbol{k}$ is parallel to the rays and therefore the ray is orthogonal to Σ. It is also worth recalling the analogy with classical mechanics and noticing that $\mathcal{L}$ is a Lagrangian and $\mathcal{S}$ an action.

Consider now a perturbation $\delta\mathcal{S}$ to $\mathcal{S}$, generated by independent perturbations $\delta\boldsymbol{x}$ and $\delta\boldsymbol{k}$ to the path $\boldsymbol{x}(t)$ and the wave number $\boldsymbol{k}$, the perturbation to the path of integration being such that it continues to start and end either at the two fixed points or on Σ, although in the latter case not necessarily at the same points. Retaining only terms linear in the perturbations, one obtains from eq. (8.7.1):

$$\delta\mathcal{S} = \int\left(\delta k_i\frac{\mathrm{d}x_i}{\mathrm{d}t} + k_i\frac{\mathrm{d}\delta x_i}{\mathrm{d}t} - \frac{\partial W}{\partial x_i}\delta x_i - \frac{\partial W}{\partial k_i}\delta k_i\right)\mathrm{d}t, \tag{8.7.3}$$

the integral being along the unperturbed path. Integrating the second term by parts yields

$$\delta\mathcal{S} = [k_i\,\delta x_i] + \int\left(\frac{\mathrm{d}x_i}{\mathrm{d}t} - \frac{\partial W}{\partial k_i}\right)\delta k_i\,\mathrm{d}t - \int\left(\frac{\mathrm{d}k_i}{\mathrm{d}t} + \frac{\partial W}{\partial x_i}\right)\delta x_i\,\mathrm{d}t, \tag{8.7.4}$$

where the square brackets denote the contribution from the perturbation at the end points. Since the path of integration is along a ray, the first term vanishes, either because $\delta\boldsymbol{x} = 0$ at the end points or because $\delta\boldsymbol{x}$ lies in Σ and $\boldsymbol{k}$ is orthogonal to Σ. Moreover, the integrals both vanish in view of the ray equations (8.3.5) and (8.3.6). Consequently

$$\delta S = 0. \tag{8.7.5}$$

The action S is stationary with respect to the perturbations. This result is valid whether or not W depends explicitly on t. (It is evident that it holds also if the ray path were to start and end on different surfaces of constant phase.)

My interest is when the background state is independent of time: $\partial W/\partial t = 0$, so that, according to eq. (8.3.8), ω is constant along a ray path, and

$$\omega = W(\boldsymbol{k}, \boldsymbol{x}). \tag{8.7.6}$$

Now consider a perturbation $(\delta\boldsymbol{x}, \delta\boldsymbol{k})$ that is constrained to satisfy eq. (8.7.6). The components of the perturbation are related according to the linearized equation

$$\frac{\partial W}{\partial k_i}\delta k_i + \frac{\partial W}{\partial x_i}\delta x_i = 0. \tag{8.7.7}$$

Substituting this relation into eq. (8.7.4) yields

$$\delta S = \delta\int \boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,\mathrm{d}t = \delta\int \boldsymbol{k}\cdot\mathrm{d}\boldsymbol{x}. \tag{8.7.8}$$

Hence, in view of eq. (8.7.5):

$$\delta\int \boldsymbol{k}\cdot\mathrm{d}\boldsymbol{x} = 0. \tag{8.7.9}$$

In other words, when W is not explicitly dependent on t, the phase integral $\int \boldsymbol{k}\cdot\mathrm{d}\boldsymbol{x}$ along a ray path is stationary with respect to arbitrary perturbations in the path of integration, whose end points remain either fixed or on the same unperturbed surfaces of constant phase and in which $\boldsymbol{k}$ is perturbed such as to preserve the dispersion relation (8.7.6).

Note, in passing, that for a nondispersive wave with phase speed c, satisfying $\omega = kc$, eq. (8.7.9) can be rewritten as

$$\delta\int\frac{\mathrm{d}s}{c} = 0, \tag{8.7.10}$$

where s is distance along the ray. This is Fermat's principle.

8.7.2. The perturbed eigenfrequency

Let the perturbed dispersion relation be

$$\omega = W_0 + W_1, \tag{8.7.11}$$

where $\omega = W_0$ is the local dispersion relation for the unperturbed system and W_1 is small compared with W_0. Assume that the perturbation is not so severe as to alter the topology of the covering space of the rays. Of course, the introduction of a perturbing agent such as rotation or a magnetic field (rather than a perturbation describing the difference between two similar stellar models constructed with essentially the same physics but with different controlling parameters), might add a new class of modes, such as inertial oscillations or torsional MHD modes, whose ray topology might be quite different, but here I am addressing only the small modification to the modes that already exist.

Although the ray paths are not closed, they are assumed to be space filling within the domain of propagation. Consider, therefore, a ray path $\mathcal{C}$ in the perturbed system which passes arbitrarily close to its starting position, so that for the purpose of computing phase integrals along it, it can be considered to be closed. Since the topology of the rays is unaltered by the perturbation, corresponding to $\mathcal{C}$ is a ray path $\mathcal{C}_0$ of the unperturbed system, which can also be considered to be almost closed and which can be generated by a continuous deformation of $\mathcal{C}$. Thus the form of the Bohr–Sommerfeld quantization conditions is unchanged by the perturbation. In particular, eq. (8.4.6), where $\boldsymbol{k}$ is the wave number of the perturbed system but $\mathcal{C}$ is now the corresponding contour $\mathcal{C}_0$ of the unperturbed system, is a correct quantization condition for the perturbed system.

The quantization condition can be modified yet further. Because $\mathcal{C}_0$ is a ray path of the unperturbed system, the variational principle expressed by eq. (8.7.9) holds (there being no boundary terms to worry about), and $\boldsymbol{k}$ may be modified by arbitrary small perturbations provided the dispersion relation (8.7.11) is preserved. Finally, because the resulting wave vector must still satisfy curl $\boldsymbol{k} = 0$, the contour may be distorted further at will, as before.

It is evident that there is a great deal of flexibility in how one perturbs the phase integrals. It is prudent to adopt a procedure that leads to integrals that are easy to evaluate. For example, when perturbing a phase integral such as eq. (8.4.10) which leads to an expression for the eigenfrequency, it is expedient to retain the zero-order horizontal components of the wave number, which are given by eqs. (8.5.1) and (8.5.2), leaving the vertical component to be calculated from them and the perturbed dispersion relation (8.7.11). This is the procedure that is used in all the examples discussed below.

8.7.3. Sound-speed perturbations

Perhaps the conceptually simplest example is a perturbation of the sound speed. I will assume the perturbation, δc, to be substantial only deep in the star, so that it is not necessary to worry about related perturbations to ω_c.

As I explained above, aspherical sound-speed perturbations refract rays out of the planes of propagation of the unperturbed spherically symmetrical configuration. Typically a ray will sample the whole of the three-dimensional region of propagation, being genuinely space filling. However, locally a ray is nearly planar, and lies close to one of the space-filling plane rays of the unperturbed star. Its path may therefore be considered to be generated by the planar ray path of the unperturbed system, but with the plane of propagation (OQP in figs. 12 and 13) itself rotating about two independent axes. In general, the ray samples the whole of the region of propagation for a given mode before returning to the vicinity of the point from which the path of the phase integral began. The perturbed phase integral determining the eigenfrequency is therefore an average of the integral (8.4.10), this average being over all polar angles ϕ'' in the unperturbed plane of propagation and over all permissible orientations of that plane.

Calculating the extent of the region of propagation and how it is sampled is a difficult task for a general aspherical perturbation. It is the geometrical analogue of the degenerate perturbation theory discussed in section 7.2. I will therefore consider only perturbations that are axisymmetric about the z-axis in fig. 12. In that case the plane of propagation rotates only about that axis, and as it rotates the background state in that plane remains unchanged. Consequently, the integral over the ray is equivalent to averaging eq. (8.4.10) over only the polar angle, ϕ''. After setting $c(r,\mu) = c_0(r) + \Delta c(r,\mu)$ and $\omega = \omega_0 + \Delta\omega$, and dropping the subscript zero of the zero-order terms, the quantization condition (8.4.10) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi'' \int_{r_1}^{r_2} \left(\frac{(\omega+\Delta\omega)^2 - \omega_c^2}{(c+\Delta c)^2} - \frac{L^2}{r^2}\right)^{1/2} \mathrm{d}r = (n - \tfrac{1}{2})\pi, \qquad (8.7.12)$$

which generalizes eq. (8.5.4). It is now a straightforward matter to subtract the zero-order condition (8.5.4), retaining only terms linear in perturbed quantities, to obtain an expression for the perturbed eigenfrequency $\Delta\omega$. It is convenient also to transform the polar angle ϕ'' in the plane to $\mu = \cos\theta$ of the basic spherical polar coordinate system, using eq. (8.6.7). Noting that each value of μ is sampled twice as ϕ'' rotates through 2π, the outcome is

$$\frac{\Delta\omega}{\omega} \simeq \frac{1}{\pi S}\int_{-\mathcal{M}}^{\mathcal{M}} (\mathcal{M}^2 - \mu^2)^{-1/2}\,\mathrm{d}\mu \int_{r_1}^{r_2} \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\Delta c}{c}\frac{\mathrm{d}r}{c}, \qquad (8.7.13)$$

where

$$S = \int_{r_1}^{r_2} \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2} \frac{\mathrm{d}r}{c}. \qquad (8.7.14)$$

It is evident from the symmetry of the kernel $(\mathcal{M}^2 - \mu^2)^{-1/2}$ that only the component of $c^{-1}\Delta c$ that is symmetric about the equator, contributes to the frequency perturbation $\Delta\omega$, as was deduced earlier from the perturbation theory described in section 7.2.

It is a simple matter to show from eqs. (8.4.5), (8.6.14) and (8.5.4) that what eqs. (8.7.13), (8.7.14) express is that the relative perturbed frequency is a weighted average of $\Delta c/c$; the weight is proportional to $(\sigma v)^{-1} r^2\,\mathrm{d}r\,\mathrm{d}\mu$, where v is the magnitude of the group velocity which can be obtained from eqs. (8.3.5) and (8.3.2) and σ is the area of a tube of rays, and it is thus the relative time the wave spends in any element of volume, $\mathrm{d}V$. Moreover, this weight can be computed from the ray of the unperturbed star, which is analogous to expressing the frequency perturbation as an integral of $\Delta c/c$ weighted by eigenfunctions of the unperturbed star, as was done in section 7. The justification for the validity of this procedure is based on the variational principle in section 8.7.1, which takes the place of the variational principle in section 5.3, upon which the perturbation theory in section 7 depends.

Since the major contribution from ω_c^2 to the integrals comes from very near the surface of the star, they can be incorporated into the phase factor $\alpha(\omega)$ introduced in section 6. Let me also assume that Δc is north–south symmetric: $\Delta c(r,-\mu) = \Delta c(r,\mu)$. (Alternatively, consider Δc now to be the north–south symmetric component of the full perturbation.) In this case, bearing in mind that by assumption Δc is negligible in the surface layers, eqs. (8.7.13) and (8.7.14) reduce to

$$S\frac{\Delta\omega}{\omega} \simeq \frac{2}{\pi}\int_0^{\mathcal{M}} \mathrm{d}\mu \int_{r_1}^{R} (\mathcal{M}^2 - \mu^2)^{-1/2} \left(1 - \frac{a^2}{w^2}\right)^{-1/2} \frac{\Delta c}{c}\,\mathrm{d}\tau, \qquad (8.7.15)$$

where

$$S \simeq \int_{r_1}^{R} \left(1 - \frac{a^2}{w^2}\right)^{-1/2} \mathrm{d}\tau - \pi\frac{\mathrm{d}\alpha}{\mathrm{d}\omega}. \qquad (8.7.16)$$

In these equations, as in section 6, $a = c/r$, $w = \omega/L$, τ is the acoustical radius, defined by eq. (5.8.22), and the phase factor $\alpha(\omega)$ is given by eq. (6.4). I have already argued that α varies only weakly with ω. Indeed, in practice the term $-\pi\,\mathrm{d}\alpha/\mathrm{d}\omega$ makes a relatively small contribution to S, and can, as a first approximation, be neglected.

The analysis can easily be generalized to include the contribution from the modified buoyancy frequency, $\mathcal{N}$. It can also be generalized formally to include sound-speed perturbations that are substantial near the upper turning point, together with the associated perturbation to ω_c^2. In both cases it is prudent to adopt a distorted coordinate system of the form (7.2.1), designed to prevent perturbations in $\mathcal{N}$ and ω_c from becoming too large near the boundaries of convection zones and in the outer layers of the star. The outcome is to replace the common quantity in parentheses in eqs. (8.7.13) and (8.7.14) by cK/ω, where K is defined by eq. (5.4.8), to multiply the integrand in eq. (8.7.14) by the factor $1-(a\mathcal{N}/w\omega)^2$ and to replace $\Delta c/c$ by $(1-\omega_c^2/\omega^2)(\Delta c/c+h)+(\omega_c\Delta\omega_c-a^2\mathcal{N}\Delta\mathcal{N}/w^2)/\omega^2+(cK/\omega)^2 dh/d\ln r$, the perturbations Δc, $\Delta\omega_c$ and $\Delta\mathcal{N}$ now being evaluated at constant x.

The formulation (8.7.15), (8.7.16) lends itself to ready inversion. Note, first of all, that the integral on the right-hand side of eq. (8.7.15) depends on the quantum numbers n, l and m and the unperturbed frequency ω only in the combinations m/L and $w=\omega/L$, the reduced frequency w being related to n and ω by eq. (6.1). Moreover, S is a function of w alone. Therefore $S\Delta\omega/\omega$ is a function of m/L and w alone, and, once the sound-speed inversion of section 6 has been carried out, can be regarded as an observable quantity. Equation (8.7.15) is a double Abel integral, and can therefore be inverted, yielding $\Delta c(a,\mu)$ in terms of the data:

$$\frac{\Delta c}{c}=\frac{2a}{\pi\mathcal{A}}\frac{\partial^2}{\partial\mu\partial a}\int_0^{\mu}d\mathcal{M}\int_{a_s}^{a}dw \times\int_{a_s}^{w}da'\frac{w\mathcal{M}\mathcal{A}(a')\Delta\omega/\omega}{a'[(\mu^2-\mathcal{M}^2)(w^2-a'^2)(a^2-w^2)]^{1/2}},\qquad(8.7.17)$$

where $a_s=a(R)$,

$$\mathcal{A}(a)=-\frac{d\ln r}{da}\qquad(8.7.18)$$

and a is related to r in terms of the unperturbed frequencies according to eq. (6.9). Of course, to carry out the inversion it is necessary to be assured that the degeneracy splitting $\Delta\omega$ is actually the outcome of a sound-speed asphericity. It is also necessary to know the unperturbed frequencies. If the aspherical component of the perturbation $\Delta c(r,\mu)$ is defined in such a way that $\int_{-1}^{1}\Delta c\,d\mu=0$, any other component of the perturbation being regarded as spherically symmetric, then for l large

$$\sum_m\Delta\omega\simeq\int_{-l}^{l}\Delta\omega\,dm$$
$$\propto\int_{-1}^{1}d\mathcal{M}\int_{-1}^{1}d\mu(\mathcal{M}^2-\mu^2)^{-1/2}(1-\mathcal{M}^2)^{-1/2}\Delta c\qquad(8.7.19)$$

$$\propto\int_{-1}^{1}\Delta c\,d\mu=0.\qquad(8.7.20)$$

Equation (8.7.20) was obtained from eq. (8.7.19) by interchanging the order of integration, the first integral then being just that encountered in eq. (6.8). Thus the frequency of the unperturbed (i.e., spherically symmetric component of the) star is simply the uniformly weighted average of all the frequencies with the same n and l and varying m. This conclusion was drawn in section 7 for all modes, asymptotic or otherwise, from the relations (7.2.10) and (7.2.11).

Finally, I should point out that if Δc is expanded in even powers of μ, as in eq. (7.2.13), and if $\pi\,da/d\omega$ is neglected in the expression (8.7.16) for S, then eq. (8.7.15) becomes

$$\frac{\Delta\omega}{\omega}=\sum\tilde{Q}_{\nu lm}\int_{r_1}^{R}\mathcal{K}c_\nu\,dr,\qquad(8.7.21)$$

where $\mathcal{K}$ is given by eq. (7.1.18) and

$$\tilde{Q}_{\nu lm}=\frac{2}{\pi}\int_0^{\mathcal{M}}\frac{\mu^{2\nu}d\mu}{(\mathcal{M}^2-\mu^2)^{1/2}}=\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\mathcal{M}^{2\nu}.\qquad(8.7.22)$$

The final expression in eq. (8.7.22) is identical with the asymptotic expression (7.2.21) for $\tilde{Q}_{\nu lm}$ defined by eq. (7.2.15), and therefore eq. (8.7.21) is the asymptotic approximation to eq. (7.2.14) for large l. This establishes the equivalence of the perturbation method in this section and that in section 7. One can do so alternatively starting from eq. (7.1.17) with Δc replaced by its appropriate angular average $\overline{\Delta c}$ defined by eq. (7.2.2), and substituting the asymptotic expression (8.6.16) for P_l^m into the formula for the spherical harmonics. Provided l is large compared with the characteristic harmonic scale of the perturbation, the oscillatory contribution from the square of the spherical harmonic averages to zero, and one is left with eq. (8.7.15), the small second term in expression (8.7.16) for S again being neglected.

8.7.4. *Frequency perturbations by a deeply buried magnetic field*

I restrict attention to a deeply buried magnetic field $\boldsymbol{B}(\boldsymbol{r})$, to be assured that the Lorentz force is always small compared to the gradient of the gas pressure. Therefore I can ignore the density stratification, thereby neglecting ω_c^2. Under these conditions the local dispersion relation becomes

$$\omega^2=k^2(c^2+v_A^2\sin^2\beta),\qquad(8.7.23)$$

where v_A is the Alfvén speed and β is the angle between the magnetic field and the wave number $\boldsymbol{k}$. Once again the dispersion relation, now eq. (8.7.23), can be used to define K in the quantization condition (8.4.10), the perturbation linearized in small quantities, and appropriate spatial averages carried out. Restricting attention once more to only axisymmetric fields, the result can be written as

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\pi S}\int_{-\mathcal{M}}^{\mathcal{M}} \mathrm{d}\mu \int_{r_1}^{R} \frac{\mathrm{d}r}{c}(\mathcal{M}^2-\mu^2)^{-1/2}\left(1-\frac{a^2}{w^2}\right)^{-1/2}\frac{v_{\mathrm{A}}^2}{c^2}\sin^2\beta. \tag{8.7.24}$$

Note that this is just the contribution arising directly from the magnetic term in the dispersion relation (8.7.23). The eigenfrequency is perturbed also by the distortion of the equilibrium state, but that can be handled in the way discussed in section 8.7.3.

I first evaluate the frequency perturbations (8.7.24) separately for azimuthal and poloidal magnetic fields. For an azimuthal field defined by $\boldsymbol{b} := (\mu_0\rho)^{-1/2}\boldsymbol{B} = (0,0,b_\phi)$, where μ_0 is the magnetic permeability of vacuum,

$$\boldsymbol{k}\cdot\boldsymbol{b} = \frac{m}{r\sin\theta}\,b_\phi\,, \tag{8.7.25}$$

from which one obtains

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\pi S}\int_{-\mathcal{M}}^{\mathcal{M}} \mathrm{d}\mu \int_{r_1}^{R}\left[1-\frac{a^2}{w^2}\left(\frac{1-\mathcal{M}^2}{1-\mu^2}\right)\right] \times(\mathcal{M}^2-\mu^2)^{-1/2}\mathcal{K}\frac{b_\phi^2}{c^2}\,\mathrm{d}r, \tag{8.7.26}$$

where $\mathcal{K}$ is given by eq. (7.1.18). For the poloidal field determined by $\boldsymbol{b} = (b_r, b_\theta, 0)$,

$$\boldsymbol{k}\cdot\boldsymbol{b} = \left(\frac{\omega^2}{c^2}-\frac{L^2}{r^2}\right)^{1/2} b_r + \frac{1}{r}\left(L^2-\frac{m^2}{\sin^2\theta}\right)^{1/2} b_\theta\,, \tag{8.7.27}$$

whence

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\pi S}\int_{-\mathcal{M}}^{\mathcal{M}} \mathrm{d}\mu \int_{r_1}^{R}\frac{1}{c^2}\left\{\frac{a^2}{w^2}b_r^2+\left[1-\frac{a^2}{w^2}\left(\frac{\mathcal{M}^2-\mu^2}{1-\mu^2}\right)\right]b_\theta^2\right\} \times(\mathcal{M}^2-\mu^2)^{-1/2}\mathcal{K}\,\mathrm{d}r. \tag{8.7.28}$$

In obtaining the last integral, it must be recalled that the integral with respect to r is really to be carried out in both directions, being the integral (8.4.10) around the closed contour C' in the covering space, and consequently the cross term that arises when squaring eq. (8.7.26), integrates to zero.

For a general axisymmetric magnetic field, further cross terms are generated in the computation of $\sin^2\beta$, but it can be shown that these also integrate to zero. Consequently, the frequency perturbation is obtained by simply summing the contributions (8.7.25) and (8.7.27). Unlike the expression (8.7.15) for degeneracy splitting by aspherical sound-speed perturbations, these formulae do not lend themselves to easy inversion.

8.7.5. Rotational splitting

As in section 7.3, I assume the star to be rotating with angular velocity $\Omega(r,\mu)$ about a unique axis. Provided $l \gg 1$, curvature plays only a minor role in the local dynamics of the oscillations, and the dispersion relation (8.3.2) continues to hold essentially unmodified in a frame of reference moving with the fluid. Thus, viewed from an inertial frame, the frequency of oscillation is given by

$$\omega - m\Omega = (\omega_c^2 + c^2k^2)^{1/2}, \tag{8.7.29}$$

where, as usual, m is the azimuthal order of the mode with respect to a coordinate system whose axis is the axis of rotation. (I refrain from using the unit vector $\boldsymbol{k}$ of section 7.3 to specify this axis, for fear of confusing it with the wave number.) When this relation is substituted into the quantization condition (8.4.10) and linearized in the perturbations, the result is simply eqs. (8.7.13)–(8.7.16) with $\Delta c/c$ replaced by $m\Omega/\omega$. The qualitative difference between the two forms of splitting, as was already noted in section 7, is that $\Delta\omega$ is now an odd function of m, whereas all other forms of splitting, which arise from agents that cannot distinguish between east and west, are even functions of m. Thus from measurements of the odd component of the degeneracy splitting one can infer unambiguously the angular velocity from the formula (8.7.17), in which $\Delta c/c$ is replaced by Ω and $\Delta\omega/\omega$ is replaced by $\Delta\omega/m$.

8.8. On the averaging of (solar) frequency data

For a star like the Sun, for which thousands of frequencies have been measured and even more are foreseen, it is neither practical nor useful to publish in the scientific journals the frequencies of all the individual modes; these should be available from data banks. However, it can be informative to publish certain properties of the data. These properties would typically be expressed as certain combinations of frequencies of different modes. There has been a tendency in the past to publish simple means of solar frequencies, the averages merely being taken over some limited range of one or more of the quantum numbers n, l or m, often without a clear indication of how different frequencies were weighted. It is apparent from the foregoing discussions that that is not the most informative thing to do.

It is not always possible to select specific combinations of mode frequencies at will. Very often, particularly at high l, individual modes cannot be resolved, and certain combinations of modes are forced upon one by the technique of observation. Nevertheless, it is useful to bear in mind what mode combinations are likely to be diagnostically the most useful, since it might be possible to tailor instrumental design and analysis techniques to suit the objectives of an investigation.

For the purpose of determining the spherically symmetrical component of the stratification of the Sun, it follows from eqs. (7.2.11), (7.2.12) and (8.7.19) that what is required for each n and l are uniformly weighted averages of the frequencies over all values of m. If further averaging is to be carried out, that should be such that the result is expressed as a function of $w = \omega/L$. For the purpose of describing asphericity, the results are evidently best described in terms of the parameters w and m/L, rather than the raw quantum numbers.

In parametrizing asphericity, at the end of section 7.2 and in section 8.7 I expanded the sound speed in even powers of $\mu = \cos\theta$. These expansion functions were chosen quite arbitrarily. Indeed, earlier in section 7.2 I used spherical harmonics, which have the advantage of orthogonality. Moreover, provided the degree of the mode is much greater than the degree of the spherical harmonic describing the perturbation, at least for axisymmetric perturbations the dependence of the splitting frequencies on m/L also separates into orthogonal (Legendre) functions, as is evident from eq. (7.2.24). Thus, expressing splitting frequencies as a series of Legendre polynomials $P_\lambda(m/L)$ is mathematically both natural and convenient.

When thinking of modes in terms of interference patterns formed by waves propagating in planes, it may seem more natural to use orthogonal functions that weight the angle of inclination of the plane uniformly, i.e. $\cos[\nu\cos^{-1}(m/L)]$, where ν is an integer. These functions bear a superficial resemblance to the asymptotic representation in appendix X of the Legendre polynomial $P_{\nu-1/2}[\cos(m/L)]$ in the oscillatory region, but they are not exactly the same. Such a weighting has essentially been considered recently by T.M. Brown and C.A. Morrow, and subsequently by F. Hill, for analyzing advection of waves by a horizontal subsurface flow. The dominant component of the flow is rotation, but superposed on it could be a flow associated with giant convective cells and large-scale meridional circulation.

The method of analysis is to restrict attention to only modes with high degree l, which are observed in a relatively small patch of the surface of the Sun, within which there is a mean horizontal velocity $\boldsymbol{U}$. (The mean is considered to be weighted uniformly with respect to horizontal coordinates over the area of observation and weighted vertically with the kernel implied by eq. (7.3.10), or alternatively its asymptotic representation (7.1.8) if the velocity varies slowly with depth.) Setting up local Cartesian coordinates x, y on the surface, with, say, x increasing towards the east and y increasing towards the north, and at each instant averaging the wave disturbance perpendicular to some fixed direction inclined with an angle ψ to the x-axis and taking its Fourier transform along that direction, extracts the locally plane components of the wave field. On Fourier transforming in time, one obtains a power spectrum $P(\omega, l;\ \psi)$, where ω and l are the temporal and spatial transformed variables. Thus, ω is the temporal frequency, as usual. However, l does not have its usual meaning; it is the magnitude of the component of the locally defined dimensionless horizontal wave number along the direction of the Fourier transform, measured in units of the inverse radius of the Sun. If the Sun were spherically symmetrical, the oscillation eigenfunctions would be separable, with spherical-harmonic angular dependence of degree l, where l is a constant integer. But when asphericity is taken into account, the angular variation of the eigenfunctions is distorted, as was discussed in section 7.6, and l now varies as the position of the observed patch is changed. Regarding the oscillations simply as waves in the surface of the Sun, with dimensionless wave number $\boldsymbol{l} = (l_x, l_y) = (l\cos\psi, l\sin\psi)$, the local asymptotic dispersion relation is

$$\omega = \tilde{\omega}(l) + R^{-1}\boldsymbol{U}\cdot\boldsymbol{l}, \tag{8.8.1}$$

where R is the solar radius and the function $\tilde{\omega}(l)$ determines the dispersion relation for a spherically symmetrical solar model (with $\boldsymbol{U} = 0$) whose radial stratification is the mean radial stratification of the actual Sun averaged over the spherical angle subtended by the observed patch. Thus, e.g., asymptotically $\tilde{\omega}(l)$ is given implicitly by eq. (6.1) or eq. (8.5.4), where $a(r)$ is to be interpreted as an angular average.

The procedure now is to consider the projection of the power spectrum onto a surface of constant ω. If the Sun were spherically symmetrical, the power would be a series of concentric circular rings about $\boldsymbol{l} = 0$ whose radii, l, decrease with the order n of the modes. Because only a patch of the Sun is observed, individual modes cannot be isolated; the power from the contributing modes merges, forming continuous rings with finite thickness. The velocity $\boldsymbol{U}$ causes both a displacement and a distortion of those rings, which can be parametrized by a radial variation $\delta l(\psi)$, as illustrated in fig. 14. Since the influence of the advection on the wave is small, the value of δl can be calculated by expanding $\tilde{\omega}$ in a power series in $\delta l = l - l_0$ about the value l_0 that l would have had for modes with frequency ω if $\boldsymbol{U}$ were zero:

$$\frac{d\tilde{\omega}}{dl}\delta l + \frac{1}{2}\frac{d^2\tilde{\omega}}{dl^2}(\delta l)^2 + \cdots + R^{-1}Ul\cos(\psi - \psi_0) = 0, \tag{8.8.2}$$

where ψ_0 defines the inclination of $\boldsymbol{U}$ to the x-axis,

$$\boldsymbol{U} =: (U\cos\psi_0, U\sin\psi_0), \tag{8.8.3}$$

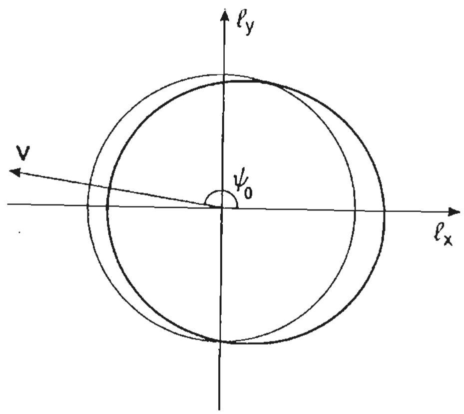


Fig. 14. Ring diagram produced by a flow with horizontal component U. Shown is the ring of the power produced by high-degree modes with the same order at a fixed frequency. The thin circle centred at the origin is where the ring would have been had U been zero.

and the derivatives of $\tilde{\omega}$ are evaluated at $l = l_0$. Equation (8.8.2) can be solved for δl, yielding

$$\delta l(\psi) = -\sum_{\nu} \Delta_\nu \cos \nu(\psi - \psi_0), \tag{8.8.4}$$

where

$$\Delta_0 = \frac{1}{2} l_0 \left(\frac{d\tilde{\omega}}{dl}\right)^{-2} \left[\frac{1}{2} l_0 \left(\frac{d\tilde{\omega}}{dl}\right)^{-1} \frac{d^2\tilde{\omega}}{dl^2} - 1\right] \frac{U^2}{R^2} + O\left(\frac{l_0^4 U^4}{R^4}\right),$$

$$\Delta_1 = l_0 \left(\frac{d\tilde{\omega}}{dl}\right)^{-1} \frac{U}{R} + O\left(\frac{l_0^3 U^3}{R^3}\right),$$

$$\Delta_2 = \Delta_0 + O\left(\frac{l_0^4 U^4}{R^4}\right), \dots \tag{8.8.5}$$

Thus, one could consider estimating l_0 and the coefficients Δ_ν by, e.g., maximizing the cross correlation between the power and a function $\mathcal{P}(l, \psi, \sigma)$, where $\mathcal{P}$ is perhaps a Gaussian ring with standard deviation σ (in l) centred about $l = l_0 + \delta l(\psi)$. By analyzing rings with different frequencies one can obtain the derivatives of $\tilde{\omega}$, and hence determine U, which, asymptotically, should be a function of $w = \omega/L$ alone. Bearing in mind that U is an average over depth, these averages can in principle be inverted, yielding the radial variation of the horizontal velocity, averaged horizontally over the area defined by the observed patch.

One can perform a similar analysis with global modes. If the frequencies ω of individual modes with the same order n can be isolated, one can construct a continuous variable $L(\psi)$ at constant frequency ω by interpolation between modes. It is important to realize that this procedure is conceptually quite different from the one described above, because L is now an interpolant of the global parameter $L_0 = l + \frac{1}{2}$ (where l is the degree of the corresponding oscillation in a spherically symmetrical solar model to which the solar mode would tend continuously as the asphericity were imagined to be reduced to zero), which characterizes a pure mode. It does not take the distortion of the mode into account and it does not vary across the surface of the star. In this case $\psi = \cos^{-1}(m/L)$, which is also obtained by interpolation. Expanding the eigenfrequency equation in powers of $\delta L = L - L_0$, as in eq. (8.8.4), yields, for an axisymmetric star rotating with angular velocity Ω and with sound-speed asphericity Δc,

$$m\langle\Omega\rangle + \omega\langle c^{-1}\Delta c\rangle = -\frac{d\tilde{\omega}}{dL}\delta L - \frac{1}{2}\frac{d^2\tilde{\omega}}{dL^2}(\delta L)^2 + \cdots, \tag{8.8.6}$$

where $\tilde{\omega}$ is the multiplet frequency (i.e. uniformly weighted average of the (singlet) frequencies of modes with the same order and degree over all azimuthal orders m), and the angular brackets denote the appropriate spatial averages determining the splitting frequencies. Since rotational splitting is an odd function of $\theta_0 = \frac{1}{2}\pi - \psi$ and the splitting due to the sound-speed asphericity is even, the two contributions can be separated, and the result inverted to obtain $\Omega(r, \mu)$ and $\Delta c(r, \mu)$ separately. In particular, if n and l are large enough for the asymptotic analysis to be valid, the angular brackets represent an average weighted with the double Abel kernel, as in eqs. (8.7.15), (8.7.16), which can then be inverted using eq. (8.7.17) and its analogue for the angular velocity.

9. Concluding remarks

The basic theory outlined in these lectures should be sufficient to equip one with the necessary expertise to develop techniques for recognizing particular patterns in oscillation eigenfrequencies that characterize certain stellar properties. Usually these techniques are based on asymptotic theory, which provides invaluable analytical formulae which depend on the equilibrium state of the star. In the case of the Sun, the simple polytropic representation (5.7.6) for high-degree p modes and Tassoul's formula (5.8.31) for low-degree p modes were the first to be used, to make deductions about the stratification of the convection zone and to calibrate the quantities A and ω_0 defined by eqs. (5.8.28) and (5.8.30). Tassoul's formula is likely to be of considerable importance to asteroseismology when stellar acoustic spectra become available, since it are only the low-degree modes that can be detected in distant stars.

As an illustration of how the theory that I have described can be used to search for some property of the background state, let us consider what might be the signature in the p-mode frequency spectrum of a steady axisymmetric jet in angular velocity, confined to some very narrow range in both latitude and depth near $\cos\theta =: \mu = \mu_0$ and $r = r_0$. We might represent it by the function $\Omega_0\,\delta(r-r_0)\,\delta(\mu-\mu_0)$, where Ω_0 is a constant and $\delta(x)$ is the Dirac delta function. For high-frequency p modes the effect of advection by rotation is to split the degeneracy by an amount $\Delta\omega$ which is approximately equal to $m\langle\Omega\rangle$, the angular brackets denoting a mean, weighted by the kinetic energy density of the unperturbed mode. All that is required now is to estimate this mean, which can be accomplished with the help of asymptotic analysis. Provided the ring with coordinates (r_0, μ_0) lies well inside the mode's region of propagation, the unperturbed displacement $\boldsymbol{\xi}$ of the oscillation in the jet is dominated by its vertical component ξ. Therefore the horizontal component η can be ignored for the jet, which is equivalent to ignoring L^2c^2/ω^2r^2 compared with unity in expression (8.5.4), defining the vertical component, K, of the wave number. Using the expressions for ξ and η derived from the equations of motion (3.6), (3.10) and (3.11), the definition of Ψ immediately preceding eq. (8.1.8) and the asymptotic relations (8.6.15) and (8.6.16), or alternatively from eqs. (5.1.1), (A10.10) and (5.8.7), (5.8.8) suitably simplified by ignoring $\mathcal{N}^2$ and approximating u by $\omega r^{-1}\rho^{1/2}$, one obtains

$$m^{-1}\Delta\omega \sim (\pi c_0 S)^{-1}\Omega_0[1-\sin 2(\omega\bar{\tau}-\alpha\pi)]Q\,, \tag{9.1}$$

where

$$Q(l,m) = (\mathcal{M}^2-\mu_0^2)^{-1/2} \times\left\{1+\sin\left[2L\cos^{-1}\left(\frac{\mu_0}{\mathcal{M}}\right)-2m\cos^{-1}\left(\frac{\mu_0}{\mathcal{M}}\sqrt{\frac{1-\mathcal{M}^2}{1-\mu_0^2}}\right)\right]\right\}, \tag{9.2}$$

with $c_0 = c(r_0)$, $L = l+\frac{1}{2}$, $\mathcal{M}$ is given by eq. (8.6.8), S by eq. (8.7.16) and $\bar{\tau}$ is the acoustical depth $T-\tau := \tau(R)-\tau(r)$ beneath the surface $r = R$ of the star. In deriving this equation I have used the polytropic approximation for the outer layers of the star, as in section 5.8, to express the phase integral directly in terms of the acoustical depth $\bar{\tau}$ and the phase shift α, which is half the effective polytropic index in the vicinity of the upper turning point.

The resulting equations provide the signature that was sought. One sees in eq. (9.1) that the frequency splitting is oscillatory with respect to the mean frequency, ω, of the multiplet, with "frequency" $2\bar{\tau}$. A measurement of that "frequency" therefore immediately determines the depth of the jet. Moreover, a moment's thought makes one realize that the presence of such an oscillatory feature

in the frequency splitting is actually indicative of a degeneracy-splitting component of the structure of the star that is confined in radius within a region that is thin compared with the vertical wavelength of the mode; the antisymmetry with respect to m indicates that the splitting agent must be rotation. The dependence on l and m given by eq. (9.2) is rather more complicated, but can evidently be used both to recognize confinement in latitude and to determine μ_0.

A word is in order about the meaning of the level $r = R$ which I have called the surface of the star. On the whole I have regarded it to be the place at which I apply boundary conditions, and since these lectures have been devoted mainly to the oscillations of the interiors of stars, and not their visible atmospheres, I have usually had in mind the interface between the optically dense interior and the atmosphere. There have been occasions, such as in the previous paragraph, at which I have invoked the complete plane-parallel polytrope to represent the outer layers of the star. This is indeed quite a good first approximation, since many stars are roughly polytropic immediately beneath their photosphere; provided the frequency of the mode is well below the acoustical cutoff frequency of the atmosphere, so that the transition to atmospheric stratification occurs well in the evanescent region for the mode, the detailed structure of the atmosphere is unimportant for the gross dynamics of the mode, and $r = R$ is then best taken to be the level at which, if c^2 were extrapolated linearly outwards from the polytropic interior, it would vanish. In the Sun, e.g., this occurs roughly 1000 km above the photosphere. With this extrapolation, one can meaningfully assign a reasonably precise definition to the acoustical depth $\bar{\tau}$.

In reality the surface layers of stars are not polytropic. Deviations are brought about mainly by ionization of hydrogen and helium, which causes sharp variations in γ (see fig. 1) and even sharper variations in ω_c^2, sometimes on length scales comparable with or shorter than the characteristic wavelength associated with the vertical variation of the oscillation eigenfunctions. For this reason the asymptotic analysis discussed in these lectures is not as accurate in the surface layers as one might like; further study of the properties of the oscillations in the superficial layers is therefore certainly needed.

There are, however, even more serious problems associated with the surface layers that need to be addressed. The most obvious are the nonadiabatic effects and the interaction with convection. To keep matters simple, I have addressed neither of these issues in these lectures. Nevertheless, it is important to appreciate their significance. Nonadiabaticity is brought about by radiative and convective energy transfer. Radiative transfer is important only in and immediately beneath the atmosphere, where also the scale heights are very much less than the stellar radius and the horizontal variation of the oscillations is of comparatively little importance. A study by Christensen-Dalsgaard and Frandsen (1983) of radial modes has shown that the dynamical effects of radiative transfer can be represented quite well in the Eddington approximation, which reduces the complicated integro-differential equations to relatively simple differential equations; we can safely assume that this approximation will work well for nonradial modes too, at least when the degree l is not large.

One does not expect nonadiabatic processes to modify the frequencies of high-degree f modes. The flow in these oscillations arranges itself such that the Lagrangian pressure perturbation δp vanishes; their is no compression nor rarefaction, and consequently the intrinsic compressibility of the gas is irrelevant to the dynamics. That is why the oscillation frequency is independent of the state of the gas. The principal effect of heat gains and losses is to influence the relation between pressure and density as they vary with the motion; but since the frequency does not depend on that relation it cannot be affected by the heat exchange. Of course, the relative magnitudes of and the phase relations between, e.g., velocity and temperature, are affected.

The effects of convection are more difficult to take into account. There is a nonadiabatic process, brought about by modulation of the convective heat flux by the oscillations. There is also a direct contribution to the momentum flux, via the Reynolds stresses. To describe both these phenomena requires a reliable time-dependent theory of convection, which is lacking. In addition to these, the spatial inhomogeneities and the intrinsic temporal variation of the convection modify the propagation speed and scatter different acoustic and gravity waves into each other, contributing to their growth or decay. Brown (1984) has recently discussed an aspect of this phenomenon, arising from the difference between acoustic Doppler shifts associated with upward and downward fluid motion, which tends to decrease the mean propagation speed and thereby diminish the oscillation frequencies. Temperature fluctuations associated with the convection behave similarly. It is important to realize, however, that Brown's discussion is principally didactic, and is intended to be no more than illustrative of one of the processes that is taking place; his estimated contribution to the frequency shift should not be used to estimate the entire influence of convection on oscillation frequencies without taking due account of the modification to the reflection properties associated with the critical cutoff, which is of the same order of magnitude. Convective inhomogeneities influence the dynamics of f modes in addition to the p modes, but perhaps to a lesser extent. Internal g modes are unlikely to be strongly affected, since the inhomogeneities are substantial only in the upper convective boundary layer, which lies in their evanescent tails.

Magnetic fields can also be dynamically significant in the outer layers of stars. In the Sun magnetic fields modify the convective flow, at least in the region of granulation, and thereby modify the influence convection has on the eigenfrequencies. It can also have a direct effect on the dynamics of the oscillations via the Lorentz force, as I have discussed in sections 7.4 and 8.6.4. In stars with magnetic cycles,

the oscillation frequencies are therefore expected to be modulated by the cycle. In the Sun the amplitude of this modulation is small. In stars with more substantial magnetic inhomogeneities, however, such as rapidly oscillating Ap stars, the magnetic field is likely to have a significant dynamical influence on the control of the oscillations that are excited, either directly through the perturbed Lorentz force or, more likely, indirectly through the magnetic modification to the thermal stratification of the background state. There is also a class of magnetic modes whose very existence depends on the presence of a magnetic field, which have received very little attention, but which might be significant in some stars. Indeed, it has been suggested that it is these, and not the more fashionable turbulent dynamo, that actually control stellar cycles.

Acknowledgements

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Appendix 1. The plane-parallel envelope

To first approximation the thin outer layers of a star can be represented by a plane-parallel envelope in hydrostatic equilibrium under constant gravitational acceleration g, satisfying the equation

$$\frac{\mathrm{d}p}{\mathrm{d}z} = g\rho, \tag{A1.1}$$

where z is the depth beneath some reference level.

The polytropic interior: if p and ρ satisfy the polytropic relation with constant index μ:

$$p = K_0 \rho^{1+1/\mu}, \tag{A1.2}$$

where K_0 is a constant, eq. (A1.1) admits the solution

$$p = p_0 \left(\frac{z}{z_0}\right)^{\mu+1}, \qquad \rho = \rho_0 \left(\frac{z}{z_0}\right)^{\mu}, \tag{A1.3}$$

where

$$(\mu + 1)\, p_0 = g \rho_0 z_0 \tag{A1.4}$$

and z_0 is simply an arbitrary scaling constant. The sound speed, c, satisfies

$$c^2 = \frac{\gamma p}{\rho} = c_0^2 \frac{z}{z_0}, \tag{A1.5}$$

provided γ is constant, where

$$c_0^2 = \frac{\gamma g z_0}{\mu + 1}. \tag{A1.6}$$

The density and pressure scale heights are, respectively,

$$H := \left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}z}\right)^{-1} = \frac{z}{\mu}, \tag{A1.7}$$

$$H_p := \left(\frac{\mathrm{d}\ln p}{\mathrm{d}z}\right)^{-1} = \frac{z}{\mu + 1}. \tag{A1.8}$$

The so-called *complete polytrope* extends downwards from $z = 0$.

The isothermal atmosphere: assume the perfect gas law

$$p = \frac{\mathcal{R}\rho T}{\mu_0}, \tag{A1.9}$$

where μ_0 is the mean molecular mass (not to be confused with the polytropic index of the envelope) and $\mathcal{R}$ the gas constant. Here T is constant, and so is the sound speed:

$$c^2 = \gamma\frac{p}{\rho} = \gamma\frac{\mathcal{R}T}{\mu_0}, \tag{A1.10}$$

provided μ_0 and γ are assumed to be constant. Using eq. (A1.9) to substitute for p in the hydrostatic equation (A1.1) yields

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{\rho}{H}, \tag{A1.11}$$

where

$$H = \frac{\mathcal{R}T}{\mu_0 g} = \frac{c^2}{\gamma g}, \tag{A1.12}$$

and I am now using the upward radial coordinate r. The solution is:

$$\rho = \rho_0\,\mathrm{e}^{(R-r)/H}, \qquad p = gH\rho_0\,\mathrm{e}^{(R-r)/H}, \tag{A1.13}$$

where ρ_0 is the density at $r = R$. The quantity H is now the scale height of both density and pressure, and is much less than the radius of the star. The major proportion of the atmosphere is therefore confined to a very thin shell, which justifies the plane-parallel approximation.

It is sometimes convenient to model the surface layers of a star by matching an isothermal atmosphere onto an incomplete polytrope represented by eq. (A1.3) for $z > z_0$. If both the pressure and temperature (and hence density) are continuous at the interface at $z = z_0$, where $r = R$, say, then $p_0 = gH\rho_0$, from which follows that $z_0 = (\mu + 1)H$. However, a more realistic representation is obtained by permitting a temperature discontinuity, to take into account the rapid variation in the superadiabatic convective boundary layer.

The effect of the small variation in the gravitational acceleration that is actually present, can easily be taken into account by expanding the solution about eq. (A1.13). Neglecting the self-gravity of the atmosphere, to leading order in the corrections the solution in the isothermal atmosphere is

$$\rho = \rho_0\psi, \qquad p = g_0 H\rho_0\psi, \tag{A1.14}$$

where

$$\psi = \mathrm{e}^{(R-r)/H}\left(1 + \frac{(r-R)^2}{RH} + \cdots\right), \tag{A1.15}$$

$g_0 = g(R) = GM/R^2$ and H is now the scale height evaluated at $r = R$.

It is useful to record expressions for the acoustical cutoff frequency, ω_c, defined by eq. (5.4.9), and the buoyancy frequency, defined by eq. (5.1.6). In the polytropic interior (A1.3),

$$\omega_c^2 = \frac{\mu(\mu+2)\gamma g}{4(\mu+1)z} = \frac{\mu(\mu+2)c_0^2}{4z_0 z}, \tag{A1.16}$$

$$N^2 = \left(\mu - \frac{\mu+1}{\gamma}\right)\frac{g}{z}. \tag{A1.17}$$

In the isothermal atmosphere (A1.13),

$$\omega_c = \frac{c}{2H} = \frac{\gamma g}{2c}, \tag{A1.18}$$

$$N^2 = \left(1 - \frac{1}{\gamma}\right)\frac{g}{H} = (\gamma - 1)\frac{g^2}{c^2}. \tag{A1.19}$$

The critical acoustic frequency, defined by eq. (4.8.7), in the plane-parallel limit, is

$$\omega_c = \frac{c_0}{2}\left(\frac{\mu^2-1}{z_0 z}\right)^{1/2} \tag{A1.20}$$

in the polytropic interior; in the isothermal atmosphere it is again given by eq. (A1.18). If the spherical correction is included, as in eq. (A1.14), in the isothermal atmosphere eq. (4.8.7) becomes

$$\omega_c \simeq \frac{c}{2H}\left(1 - \frac{4(2-\gamma)H}{\gamma R} - 2\,\frac{r-R}{R}\right). \tag{A1.21}$$

Appendix 2. Reality of eigenfrequencies below the critical cutoff

Equation (4.5.1) is

$$I(\xi,\xi^*)\omega^2 = K(\xi,\xi^*) - \gamma p r^4 \tilde{\kappa}(\omega)\xi^*\xi\big|_{r=R}, \tag{A2.1}$$

where $\tilde{\kappa}$ is given by eq. (4.2.13) with the negative sign. This may be rewritten in the form

$$a\omega^2 - b = \lambda(\omega_c^2 - \omega^2)^{1/2}, \tag{A2.2}$$

where $\lambda = \pm 1$, ω_c^2 is evaluated at $r = R$ and

$$a = \Lambda I, \qquad b = \Lambda K - \frac{c}{2H}\bigg|_{r=R}, \qquad \Lambda = \frac{c}{\gamma p R^4 \xi^* \xi}\bigg|_{r=R}. \tag{A2.3}$$

The coefficients a and b are real, with $a > 0$. Setting $\omega^2 = x + iy$, with x and y real, the real and imaginary part of the square of eq. (A2.2) are

$$a^2(x^2 - y^2) - (2ab - \lambda^2)x + b^2 - \lambda^2\omega_c^2 = 0 \tag{A2.4}$$

$$(2a^2 x - 2ab + \lambda^2)y = 0. \tag{A2.5}$$

It follows from eq. (A2.5) that

$$x = \frac{b}{a} - \frac{\lambda^2}{2a^2}, \quad \text{if } y \neq 0. \tag{A2.6}$$

Moreover, eq. (A2.4) can be solved formally for x, giving

$$x = \frac{b}{a} - \frac{\lambda^2}{2a^2} \pm \frac{1}{2a^2}(\lambda^4 - 4\lambda^2 ab + 4\lambda^2 a^2 \omega_c^2 + 4a^4 y^2)^{1/2}. \tag{A2.7}$$

Thus, if $y \neq 0$, the two equations (A2.6) and (A2.7) must be identical; the square root in eq. (A2.7) must therefore vanish, and hence

$$y^2 = (-\lambda^4 + 4\lambda^2 ab - 4\lambda^2 a^2 \omega_c^2)/4a^4, \tag{A2.8}$$

from which follows that

$$\omega_c^2 < \frac{b}{a} - \frac{\lambda^2}{4a^2} =: \omega_{c0}^2, \tag{A2.9}$$

since y is real.

If the inequality (A2.9) is not satisfied, then $y = 0$ and ω^2 is real. Then eq. (A2.2) becomes

$$x = \frac{b}{a} + \frac{\lambda}{a}(\omega_c^2 - x)^{1/2}. \tag{A2.10}$$

This is possible only when $\omega^2 < \omega_c^2$. Comparing this with eq. (A2.7) requires choosing the positive square root in eq. (A2.7). However, the two expressions are still not necessarily equivalent; when $\omega_c^2 < b/a$ the solution (A2.7) with $y = 0$ satisfies eq. (A2.10) with $\lambda = -1$. For $\lambda = +1$, which applies to eq. (A2.1), ω_c^2 must exceed b/a; at the critical value b/a, $x = \omega_c^2$. Otherwise $x < \omega_c^2$.

If, on the other hand, the inequality (A2.9) is satisfied, so that ω^2 is complex, then x is given by eq. (A2.6) and is independent of ω_c^2. The real part, ω_R, of ω is related to x and y by

$$\begin{aligned}\omega_R^2 &= \tfrac{1}{2}x + \tfrac{1}{2}\sqrt{(x^2 + y^2)} \\ &= \tfrac{1}{2}x + \tfrac{1}{2}[x^2 + a^{-2}(x + \tfrac{1}{4}a^{-2} - \omega_c^2)]^{1/2},\end{aligned} \tag{A2.11}$$

which implies that $\omega_R = x$ when $\omega_c^2 = \omega_{c0}^2 = x + \frac{1}{4}a^{-2}$. It is now a simple matter to show from this expression and the discussion in the previous paragraph that if $\omega_c^2 < \omega_{c0}^2$ then $\omega_c^2 < x$ and $\omega_R > \omega_c$.

Appendix 3. The Roche stellar model and its radial pulsations

This model approximates the structure of a highly centrally condensed stellar model with mass M, luminosity L and radius R. The stratification of the star is estimated by assuming the gravitational acceleration to be given by the assumption that the entire stellar mass is concentrated in a small core, so that hydrostatic support outside the core is determined by

$$\frac{dp}{dr} \simeq -\frac{GM\rho}{r^2}. \tag{A3.1}$$

In addition it is assumed that the stellar material satisfies the polytropic relation

$$p = K_0\rho^\Gamma, \tag{A3.2}$$

where K_0 and Γ are constants. This would be so if the envelope were fully convective, and therefore essentially adiabatically stratified, with $\Gamma = \gamma$, provided one could assume γ to be constant. It is also the case if the envelope is in radiative equilibrium, with a power-law opacity

$$\kappa = \kappa_0\rho^\lambda T^{-\nu}, \tag{A3.3}$$

such as Kramers' law ($\lambda = 1$, $\nu = 3.5$). In this case, dividing eq. (A3.1) by the radiative transport equation

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi a\tilde{c}r^2T^3}, \tag{A3.4}$$

where a is the radiation constant and $\tilde{c}$ is the speed of light, and assuming the perfect gas law (1.1.4) with μ constant, yields

$$p^\lambda\frac{dp}{dT} = \frac{16\pi a\tilde{c}GM}{3\kappa_0 L}\left(\frac{\mathcal{R}}{\mu}\right)^\lambda T^{3+\nu+\lambda}. \tag{A3.5}$$

On integration it follows that p is proportional to a power of T, and hence a power of ρ, at least deep in the interior where any constant of integration of eq. (A3.5), required for satisfying suitable boundary conditions at the surface, is negligible. For simplicity I take the approximate opacity law (A3.3) with $\lambda = 1$, $\nu = 3$; then eqs. (A3.1), (A3.2) can be integrated, yielding

$$\frac{T}{T_0} = \left(\frac{r}{r_0}\right)^{-1}, \quad \frac{\rho}{\rho_0} = \left(\frac{r}{r_0}\right)^{-3}, \quad \frac{p}{p_0} = \left(\frac{r}{r_0}\right)^{-4}, \tag{A3.6}$$

where T_0, ρ_0 and p_0 are the temperature, density and pressure at some arbitrary reference radius r_0. This solution can be used, e.g., as a crude representation of a red-giant envelope, in which the sound speed is given by

$$c = c_0\left(\frac{r}{r_0}\right)^{-1/2}. \tag{A3.7}$$

Setting $r_0 = R$, for convenience, and substituting the second and third of the expressions (A3.6) into the hydrostatic equation (A3.1) to relate p_0 and ρ_0 yields

$$c_0^2 = \frac{\gamma p_0}{\rho_0} = \tfrac{1}{4}\gamma R^2\omega_0^2, \tag{A3.8}$$

ω_0 being the characteristic pulsation frequency defined by eq. (4.2.6).

Adiabatic radial pulsations in the Roche model satisfy eq. (4.7.1). This is transformed into the Bessel equation

$$\frac{d^2}{dx^2}(x^{-1/3}\xi) + \frac{1}{x}\frac{d}{dx}(x^{-1/3}\xi) + \left(1 - \frac{\lambda^2}{x^2}\right)x^{-1/3}\xi = 0 \tag{A3.9}$$

by the substitution

$$x = \frac{2\omega R}{3c_0}\left(\frac{r}{R}\right)^{3/2}, \tag{A3.10}$$

$$\lambda^2 = \frac{4}{9}(3\gamma - 4)\frac{R^2\omega_0^2}{c_0^2} + \frac{1}{9} = \frac{1}{9\gamma}(49\gamma - 64). \tag{A3.11}$$

The eigenfunction is the solution to eq. (A3.9) that is regular at the origin (one cannot apply the boundary condition (4.2.4) since the representation (A3.6) of the equilibrium model is not valid all the way to $r = 0$), i.e.

$$\xi = r^{1/2}J_\lambda\left[\frac{2\omega R}{3c_0}\left(\frac{r}{R}\right)^{3/2}\right]. \tag{A3.12}$$

Note that for small radii $\xi \propto r^{(3\lambda+1)/2}$, consistent with Epstein's (1950) finding that the fundamental eigenfunction of a giant star is small near the centre. Substituting the solution (A3.12) into the approximate surface boundary condition (4.8.16) yields the eigenvalue equation

$$\frac{2\omega R}{3c_0}J'_\lambda\left(\frac{2\omega R}{3c_0}\right) + \frac{1}{3\gamma}\left(7\gamma - 8 - 2\frac{\omega^2}{\omega_0^2}\right)J_\lambda\left(\frac{2\omega R}{3c_0}\right) = 0, \tag{A3.13}$$

where the prime denotes differentiation with respect to the argument. The fundamental solution to eq. (A3.13) is $\omega = \omega_1 \simeq 1.6\omega_0$ when $\gamma = \frac{5}{3}$. At high order, n, the solution can be obtained from the asymptotic expansion of the Bessel function and its derivative:

$$\omega = \omega_n \sim \bar{\omega}_0\left[n + \frac{1}{2}\lambda - \frac{1}{4} + \frac{256 + 5\gamma}{72\pi^2\gamma\left(n + \frac{1}{2}\lambda - \frac{1}{4}\right)} + O(n^{-3})\right], \tag{A3.14}$$

where

$$\bar{\omega}_0 = \frac{3\pi c_0}{2R} = \pi\left(\int_0^R \frac{dr}{c}\right)^{-1}, \quad (A3.15)$$

is the characteristic frequency defined by eq. (A3.15), the second part of eq. (A3.15) being obtained by evaluating the integral with the help of eq. (A3.7). From eqs. (A3.15), (A3.7) and (A3.8) one obtains $\bar{\omega}_0 = \frac{3}{4}\pi\sqrt{\gamma}\,\omega_0$. Thus $\omega_0 \propto M^{1/2}R^{-3/2}$ is the natural frequency unit for modes of all orders. Since the low-frequency boundary condition (4.8.16) was used to derive the eigenvalue equation (A3.13), approximation (A3.14) of the eigenfrequencies, valid for $n \gg 1$, breaks down once ω becomes comparable with the critical cutoff frequency in the atmosphere. For such high frequencies the more accurate condition (4.8.15) should be used.

Appendix 4. JWKB expansion of the damped oscillator equation

We seek an approximate solution of the equation with constant coefficients:

$$\frac{d^2y}{dx^2} + 2\kappa\frac{dy}{dx} + K^2y = 0, \quad 0 \leqslant x \leqslant 1, \quad (A4.1)$$

where $K^2 \gg 1$ and $\kappa = O(1)$, subject to the boundary conditions

$$y = 0 \quad \text{at } x = 0 \text{ and } x = 1 \quad (A4.2)$$

and

$$\frac{dy}{dx} = 1 \quad \text{at } x = 0. \quad (A4.3)$$

A direct attack can be made simply by setting

$$y = A\exp\left[iK\int_0^x \psi(s)\,ds\right] \quad (A4.4)$$

and expanding the functions A and ψ in inverse powers of K:

$$A = A_0 + K^{-1}A_1 + K^{-2}A_2 + \cdots, \quad (A4.5)$$

$$\psi = \psi_0 + K^{-1}\psi_1 + K^{-2}\psi_2 + \cdots. \quad (A4.6)$$

Substituting the form (A4.4) into eq. (A4.1) yields

$$-K^2\psi^2A + iK(2\psi A' + \psi' A + 2\kappa\psi A) + A'' + 2\kappa A' + K^2A = 0, \quad (A4.7)$$

where the prime denotes differentiation with respect to x. The expansions (A4.5) and (A4.6) are then entered into eq. (A4.7) and the coefficient of each power of K is equated to zero. The leading term is

$$1 - \psi_0^2 = 0, \quad (A4.8)$$

which has the solutions

$$\psi_0 = \pm 1. \quad (A4.9)$$

The coefficient of K is

$$-2\psi_0\psi_1A_0 + 2i\psi_0(A_0' + \kappa A_0) = 0, \quad (A4.10)$$

since $\psi_0' = 0$. This relates ψ_1 to A_0. There is no additional restriction on the relation between ψ_1 and A_0 from higher-order terms in the expansion, so without loss of generality I may set $\psi_1 = 0$. Then

$$\mathcal{L}A_0 := A_0' + \kappa A_0 = 0, \quad (A4.11)$$

whose solution is

$$A_0 = \Lambda\, e^{-\kappa x}. \quad (A4.12)$$

Note that if, e.g., I had set $A_0 = \Lambda$, from eq. (A4.10) I would have had $\psi_1 = i\kappa$, and $A_0\exp(iK\int K^{-1}\psi_1\,dx)$ would not have been affected. The constant Λ will eventually be determined by the normalization condition (A4.3); in order to specify the higher-order terms in the sequence (A4.5), which satisfy inhomogeneous equations of the form $\mathcal{L}A_k = f_k$ and therefore admit complementary functions proportional to A_0, I will insist that $A_k = 0$ at $x = 0$. One can now equate to zero the coefficient of K^0:

$$-2\psi_0\psi_2A_0 + 2i\psi_0(A_1' + \kappa A_1) + A_0'' + 2\kappa A_0' = 0, \quad (A4.13)$$

which may be rewritten as

$$A_1' + \kappa A_1 = -i\Lambda(\tfrac{1}{2}\kappa^2\psi_0^{-1} + \psi_2)\,e^{-\kappa x}. \quad (A4.14)$$

Once again I have some freedom, and this time I choose $A_1 = 0$. Then

$$\psi_2 = -\tfrac{1}{2}\kappa^2\psi_0^{-1}. \tag{A4.15}$$

To this order of approximation, the solution satisfying $y = 0$ at $x = 0$ is thus

$$y = \Lambda\, e^{-\kappa x} \sin(K - \tfrac{1}{2}K^{-1}\kappa^2 + \cdots)x. \tag{A4.16}$$

Application of the remaining boundary conditions yields the eigenvalue equation

$$K - \tfrac{1}{2}K^{-1}\kappa^2 + \cdots = n\pi, \quad n = 1, 2, \ldots \tag{A4.17}$$

and the normalization

$$\Lambda = K^{-1} + \tfrac{1}{2}K^{-3}\kappa^2 + \cdots. \tag{A4.18}$$

Had I set $\psi_2 = 0$ in eq. (A4.14), the solution would have been obtained in the form

$$\Lambda\, e^{-\kappa x}(\sin Kx - \tfrac{1}{2}K^{-1}\kappa^2 x \cos Kx + \cdots), \tag{A4.19}$$

from which eqs. (A4.17) and (A4.18) also follow.

If, instead of attacking the raw problem, one first reduces eq. (A4.1) to the standard form

$$\frac{d^2\eta}{dx^2} + (K^2 - \kappa^2)\eta = 0, \tag{A4.20}$$

where $\eta = y\, e^{\kappa x}$, one obtains immediately (from the JWKB approximation!)

$$y = \Lambda\, e^{-\kappa x} \sin(K^2 - \kappa^2)^{1/2} x, \tag{A4.21}$$

from which

$$(K^2 - \kappa^2)^{1/2} = n\pi, \quad n = 1, 2, \ldots \tag{A4.22}$$

and

$$\Lambda = (K^2 - \kappa^2)^{-1/2}. \tag{A4.23}$$

Equations (A4.17) and (A4.18) follow immediately from an expansion of eqs. (A4.22) and (A4.23) in powers of κ/K, but in fact we well know that the latter pair of equations is best left untouched.

Appendix 5. Causal adiabatic oscillations of an isothermal atmosphere

For the plane-parallel isothermal atmosphere of appendix I under constant gravitational acceleration, the equation of motion (5.2.7) for adiabatic oscillation with horizontal wave number of magnitude k can be rewritten as

$$\frac{\partial^2 w}{\partial z^2} + \left[\frac{\omega^2 - \omega_c^2}{c^2} - k^2\left(1 - \frac{N^2}{\omega^2}\right)\right] w = 0, \tag{A5.1}$$

where z is the height above $r = R$,

$$w(z, t) = e^{-z/2H}\, \xi(z)\, e^{-i\omega t} \tag{A5.2}$$

and ω_c is given by eq. (4.2.14). Note that w is not the vertical component of the velocity; it is the vertical component of the displacement scaled by the square root of the equilibrium density.

We first note that the coefficient of w in eq. (A5.1) is constant, and therefore the equation admits wave-like solutions

$$w = w_0\, e^{i(Kz - \omega t)}, \tag{A5.3}$$

where

$$K = \pm\left[\frac{\omega^2 - \omega_c^2}{c^2} - k^2\left(1 - \frac{N^2}{\omega^2}\right)\right]^{1/2} \tag{A5.4}$$

is the vertical wave number. This equation may alternatively be regarded as a dispersion relation determining ω in terms of k and K. Then it is a quadratic equation for ω^2 with roots

$$\omega^2 = \omega_\pm^2 := \chi^2 c^2 \pm (\chi^4 c^4 - N^2 k^2 c^2)^{1/2}, \tag{A5.5}$$

where

$$\chi^2 = \tfrac{1}{2}(k^2 + K^2 + \omega_c^2/c^2). \tag{A5.6}$$

Let us first assume that K is real, so that the solution (A5.3) represents a propagating wave. The positive root (A5.5) is for acoustic waves, modified by the stratification; when the total wave number $(k^2 + K^2)^{1/2}$ is large, eq. (A5.5) approximates the acoustic dispersion relation $\omega^2 \simeq (k^2 + K^2)c^2$ for a uniform gas. The negative sign is for internal gravity waves; when K^2 is large the usual dispersion relation $\omega^2 \simeq k^2N^2/(k^2 + K^2)$ is recovered. In both cases the positive and negative values of ω resulting from the two possible square roots, correspond to waves whose vertical phase propagation speed, ω/K, is upwards and downwards, respectively (if K is positive).

The solution (A5.3) can be used to derive a boundary condition

$$\frac{d\xi}{dr} - \left(\frac{1}{2H} + iK\right)\xi = 0 \tag{A5.7}$$

to be applied at $r = R$ to the stellar oscillation eigenfunctions, once the causal root of eq. (A5.4) has been established.

One approach to ascertaining that root is to compute the vertical component of the group velocity from the dispersion relation (A5.5):

$$\frac{\partial\omega}{\partial K} = \frac{\omega K c^2}{2(\omega^2 - \chi^2 c^2)}, \tag{A5.8}$$

which has the same sign as ω/K for acoustic waves and the opposite sign for internal gravity waves, assuming both ω and K to be real. To begin, I take ω to be real and positive. Since the causal solution should have an outward directed group velocity, one must therefore choose the positive root of eq. (A5.4) for p modes and the negative root for g modes. Of course, if energy is propagating outwards, ω is unlikely to be real, but the solution with complex ω can be obtained by analytic continuation from the appropriate root with real ω.

When $\omega_-^2 < \omega^2 < \omega_+^2$ the mode is evanescent; in that case one would expect the causal solution to be the one for which energy density decreases upwards. I now demonstrate that if the atmospheric motion is the response to an oscillation of the star below, that is indeed the case.

Consider an initial-value problem for which the entire atmosphere is at rest when $t < 0$, and subsequently the base of the atmosphere is given an oscillatory displacement such that

$$w = W \sin\omega t \quad \text{at } z = 0,\ t \geqslant 0, \tag{A5.9}$$

where W is a constant and now $w(z,t) = \exp(-z/2H)\,\xi(z,t)$. The problem can be solved by taking the Laplace transform

$$\widehat{w}(z,p) := \int_0^\infty e^{-pt}\, w(z,t)\,dt, \tag{A5.10}$$

with $\mathrm{Re}(p) > 0$. The transformed equation of motion is

$$\frac{\partial^2\widehat{w}}{\partial z^2} - \kappa^2\widehat{w} = 0, \tag{A5.11}$$

since $w = 0$ and $\partial w/\partial t = 0$ at $t = 0$, $z > 0$. Here

$$\kappa(p) = \pm\left[\frac{p^2 + \omega_c^2}{c^2} + k^2\left(1 + \frac{N^2}{p^2}\right)\right]^{1/2} \tag{A5.12}$$

(cf. eq. (A5.1)). Equation (A5.11) is to be solved subject to the boundary conditions

$$\widehat{w} = W\int_0^\infty e^{-pt}\sin\omega t\,dt = -\frac{\omega W}{p^2 + \omega^2} \quad \text{at } z = 0 \tag{A5.13}$$

and

$$\widehat{w} \to 0 \quad \text{as } z \to \infty. \tag{A5.14}$$

The solution to the problem is then given by

$$w = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{pt}\,\widehat{w}\,dp, \tag{A5.15}$$

where $\mathcal{L}$ is the Bromwich contour extending from $-i\infty$ to $+i\infty$ to the right of all the singularities of $\widehat{w}$ in the complex p-plane.

The solution of eq. (A5.11) satisfying the condition (A5.13) is

$$\widehat{w} = -\frac{\omega W}{p^2 + \omega^2}\, e^{-\kappa z}, \tag{A5.16}$$

where, to satisfy condition (A5.14), κ is to be interpreted as that branch of the square root of the right-hand side of eq. (A5.12) whose real part is positive on $\mathcal{L}$.

The calculation of the integral (A5.15) depends on the value of t. Noting that as $|p| \to \infty$, $\kappa \sim p/c$, it follows that if $t < z/c$, $\mathrm{Re}(pt - \kappa z) < 0$ on the semi-circle $\mathcal{C}$ at infinity in the half-plane: $\mathrm{Re}(p) > 0$. Hence the integral along $\mathcal{C}$ is zero, and therefore the contour $\mathcal{L}$ can be closed by $\mathcal{C}$ without changing the value of the integral. Since the resulting contour encloses no singularity, it follows from Cauchy's theorem that $w = 0$.

When $t > z/c$, $\mathrm{Re}(pt - \kappa z) < 0$ when $\mathrm{Re}(p) \ll -1$, and it is useful to attempt to move $\mathcal{L}$ leftwards to $p = iv - \infty$, where v is real. But before doing so, it is convenient to make two cuts in the p-plane that join pairs of branch points of κ, such as those illustrated in fig. 15a. Then each branch of κ, and therefore each branch of $\widehat{w}\,e^{pt}$, is single-valued in the resulting p-plane. After moving $\mathcal{L}$ far to the left, two loops, $\mathcal{L}_1'$ and $\mathcal{L}_2'$, encircling the branch cuts are left, together with small circles about the pole of κ at the origin and about the poles of $\widehat{w}$ at $p = \pm i\omega$.

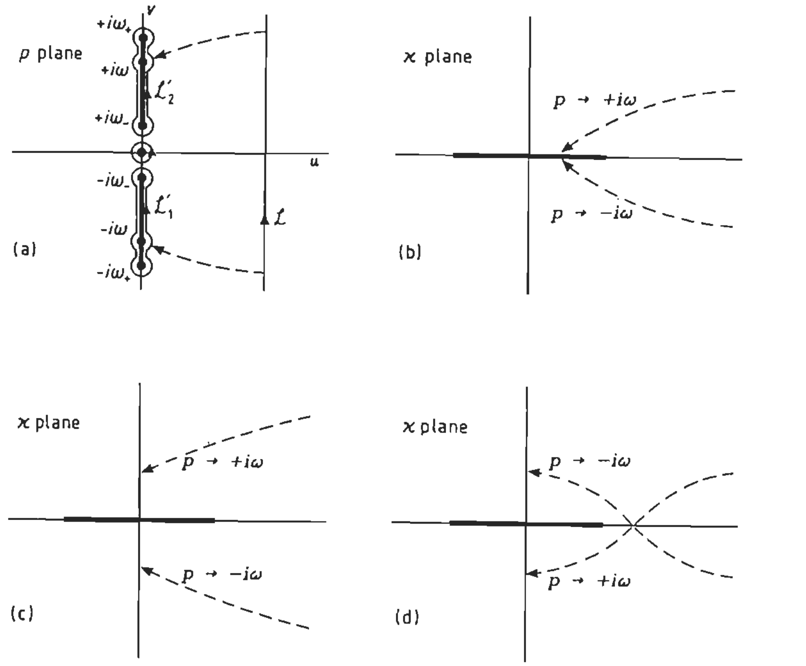

Fig. 15. (a) The complex p-plane, cut between pairs of branch points of κ, showing the Bromwich contour $\mathcal{L}$ and the three contours $\mathcal{L}'_1$, $\mathcal{L}'_2$ and the circle about the origin into which it can be deformed, for the evanescent case having $\omega_-^2 < \omega^2 < \omega_+^2$. The dashed arrows represent the movement of two of the points on $\mathcal{L}$ that approach the poles of $\widehat{w}$ as the contour is deformed. There are other paths, approaching the poles from the opposite side of the branch cut, but these have not been shown in order not to clutter the diagram excessively. The corresponding κ-plane is illustrated in (b), which shows how the branch cuts and the dashed arrows are mapped; (c) and (d) are the corresponding κ-planes for propagating acoustic waves and propagating gravity waves, respectively.

Figure 15a illustrates the evanescent case, in which the poles of $\widehat{w}$ lie on the branch cut. The contributions to the total integral can now be divided into three classes: (i) integrals around small circles, of radius ϵ, say, about the branch points and the pole of $\kappa(p)$, which are at $p = \pm i\omega_\pm$ and $p = 0$, respectively, (ii) integrals adjacent to the cut portions of the imaginary axis and (iii) integrals around the poles of $\widehat{w}$. The integrand $\widehat{w}\,e^{pt}$ is bounded on the small circles about $p = \pm i\omega_\pm$ and $p = 0$, and therefore the integrals of class (i) vanish as $\epsilon \to 0$. The integrals adjacent to the cuts represent the transient to the initiation of the oscillation of the lower boundary, producing what is sometimes called a "tail" behind the front of the propagating disturbance. The integrand has a factor e^{pt}, which becomes increasingly oscillatory as t increases, and by Riemann's lemma the integrals vanish

in the limit $t \to \infty$. Since our interest is only in selecting the appropriate root of the long-term response to the movement of the lower boundary, we may take that limit and ignore the transient. There remain only the integrals around the poles at $p = \pm i\omega$. To trace the appropriate square root in eq. (A5.12) for evaluating the residues at those poles, it is convenient to map the p-plane onto the κ-plane. The dashed paths in the p-plane from points on $\mathcal{L}$, where $\mathrm{Re}(\kappa) > 0$, to the two poles of the integrand, are drawn schematically in the complex κ-plane in figs. 15b–d. To see how they map, it is convenient to set $p = u + iv$, where u and v are real; then eq. (A5.12) can be rewritten as

$$\kappa = \left[\frac{\omega_c^2 + u^2 - v^2}{c^2} + k^2\left(1 + \frac{N^2(u^2 - v^2)}{(u^2 + v^2)^2}\right) + \frac{2iuv}{c^2}\left(1 - \frac{N^2k^2c^2}{(u^2+v^2)^2}\right)\right]^{1/2}. \tag{A5.17}$$

Figure 15b illustrates a case for the evanescent solution with $\omega^4 > N^2k^2c^2$ when, as $u \to 0$ and $v \to \pm\omega$, $\kappa^2 \to -K^2 =: \kappa_0^2$ with κ_0 real and positive. It is evident that the appropriate root is $\kappa = \kappa_0$. Then

$$w \sim W\,e^{-\kappa_0 z}\sin\omega t \quad \text{as } t \to \infty. \tag{A5.18}$$

If $\omega^4 < N^2k^2c^2$, fig. 15b still applies, at least if $\mathcal{L}$ is close to the imaginary axis in the p-plane, except that the labels on the dashed curves are interchanged; if $\mathcal{L}$ is far to the right of the imaginary axis in the p-plane the dashed curves cross on the real axis of the κ-plane, as they do in fig. 15d. The solution is still given by eq. (A5.18), with $w = 0$ where $z > ct$.

Propagating acoustic-wave solutions, for which $\kappa^2 \to -K^2 =: -\kappa_0^2$ as $p \to \pm i\omega$, with κ_0 real and positive, are illustrated in fig. 15c. Note that in this case the Bromwich integral can again be reduced to integrals around closed contours analogous to $\mathcal{L}'_1$ and $\mathcal{L}'_2$ and the circle about the origin of fig. 15a, except that now the poles at $p = \pm i\omega$ are no longer between the branch points of κ; there continues to be a transient coming from the integrals either side of the branch cut, and the long-term response can be reduced to integrals around isolated contours encircling the poles of $\widehat{w}$. Since $\omega^4 > N^2k^2c^2$, the dashed paths do not cross the real axis on their approach to the singularities of $\widehat{w}$. Consequently, $\kappa \to \pm i\kappa_0$ as $p \to \pm i\omega$, and

$$w \sim W\sin(\omega t - \kappa_0 z) \quad \text{as } t \to \infty. \tag{A5.19}$$

Internal gravity waves are treated similarly. They satisfy $\omega^4 < N^2k^2c^2$, and therefore now the dashed paths in the κ-plane to the poles of $\widehat{w}$ do cross the real axis before they reach $-\kappa_0^2$, as is illustrated in fig. 15d. Therefore $\kappa \to \mp i\kappa_0$ as $p \to \pm i\omega$, and

$$w \sim W \sin(\omega t + \kappa_0 z) \quad \text{as } t \to \infty. \tag{A5.20}$$

These results agree with the conclusions drawn earlier in this appendix from applying the radiation condition to propagating waves and by insisting that the energy density decreases upwards for the evanescent oscillations.

An interesting consequence of the condition that the energy density decreases upwards is that the f mode given by eqs. (5.6.4)–(5.6.6), can be excited from below only if $k < k_c := \frac{1}{2}H^{-1}$. If $k > k_c$, then it is the second solution (5.6.7) that is excited. It is not obvious that such a transition would occur in a real star, such as the Sun, however. In a star there is no oscillating rigid boundary in the convection zone; unless the nonlinear interactions are strong enough to prevent the exponential rise of the amplitude with depth, it would seem that f modes given by eq. (5.6.7) are unlikely to be excited to observable amplitudes. If that is so, then if f modes with $k > k_c$ are observed in the Sun, one might expect them to have been generated from above.

The initial-value problem can be solved for the adiabatic response of the atmosphere to a mode with nonzero growth rate. If eq. (A5.9) is replaced by

$$w = W\,e^{\eta t}\sin\omega t \quad \text{at } z = 0,\ t \geqslant 0, \tag{A5.21}$$

the only change is to move the poles off the imaginary axis in fig. 15a by an amount η; the points to which the dashed paths in figs. 15b–d lead are also moved off the real axis. Otherwise, aside from minor distortions of the contours $\mathcal{L}_1'$ and $\mathcal{L}_2'$, the analysis is unaltered. If $\eta^2 \ll \omega^2$, the solutions (A5.18)–(A5.20) become, respectively,

$$w \sim W\,e^{\eta t - \kappa_0 z}\sin[\omega(t - \eta\psi z)], \tag{A5.22}$$
$$W\,e^{\eta(t-\omega\psi z)}\sin(\omega t - \kappa_0 z), \tag{A5.23}$$
$$W\,e^{\eta(t+\omega\psi z)}\sin(\omega t + \kappa_0 z), \tag{A5.24}$$

as $t \to \infty$, where

$$\psi = \frac{\omega}{c^2\kappa_0}\left(1 - \frac{N^2k^2c^2}{\omega^4}\right) \tag{A5.25}$$

and $\kappa_0 = |K|$; also $w = 0$ for $z > ct$. In all cases, whether the waves are acoustic waves or gravity waves, or whether they are evanescent, the front of the disturbance propagates upwards with the speed of sound. Finally, I record that in the propagating cases (A5.23) and (A5.24) the energy density, and consequently also

the energy flux, increases upwards when $\eta < 0$. This is because the disturbance at any given height in the atmosphere has passed some reference level in the star, say the photosphere, before the disturbance at any lesser height does, and thus at a time when the pulsation amplitude in the star was greater.

Appendix 6. The oscillation of a plane-parallel polytrope supporting an isothermal atmosphere

The purpose of this appendix is to illustrate the influence of the atmosphere on the frequencies of p modes. I consider separately the modes with high and low degree.

The equations of motion, in the various forms I will need them, are given in the main text, and will be referred to explicitly when the need arises. For high-degree modes the solution must decay to zero at great depths, and for $k = 0$ I impose an artificial lower boundary condition which I will discuss later. The upper boundary condition is obtained by selecting the causal solution (cf. appendix V). I will consider only modes with frequencies below the acoustical cutoff in the atmosphere, so the appropriate solution is the one whose displacement amplitude grows the more slowly with height. On the perturbed interface between the two regions the pressure and the vertical component of the displacement must be continuous. It is straightforward to show that after linearization these conditions are equivalent to the continuity of ξ and δp on the unperturbed surface $z = z_0$.

High-degree modes are confined to the outer layers of the star, which I represent by the plane-parallel polytrope with index μ supporting an isothermal atmosphere with scale height H discussed in appendix I. For simplicity I regard the density to be continuous at the interface at $z = z_0$, so that $z_0 = (\mu + 1)H$.

In the polytrope, $z > z_0$, the quantity $\chi = \rho^{-1/2}c^{-2}\,e^{kz}\,\Psi \propto e^{kz}\operatorname{div}\boldsymbol{\xi}$ satisfies eq. (5.7.2) and has the solution (5.7.5) for which $\operatorname{div}\boldsymbol{\xi}$ and the perturbations $\delta\rho$ and δp vanish as $z \to \infty$. In the atmosphere eq. (5.4.8) reduces to

$$K^2 = \frac{1}{4H^2}\{1 - 4[\gamma^{-1}(\sigma^2 - \sigma^{-2}) + \sigma^{-2}]Hk + 4H^2k^2\}, \tag{A6.1}$$

where $k = R^{-1}L$ is the horizontal wave number. Hence,

$$\chi = \chi_0 U(-a, \mu + 2, 2kz_0)\,e^{-(\kappa - k)(z - z_0)}, \qquad z < z_0, \tag{A6.2}$$

where a is defined by eq. (5.7.3), $\kappa = \frac{1}{2}H^{-1} - K$ with $K > 0$ for the causal solution (cf. appendix V), and I have chosen the constant of proportionality so as to ensure that χ, and hence $\delta\rho$, is continuous at $z = z_0$.

The frequency is determined by requiring the continuity of the vertical component, ξ, of the displacement, which, according to eqs. (5.1.10), (5.1.11) and (5.4.6) is given by

$$\xi \propto \rho^{-1}\left(\frac{\mathrm{d}\delta p}{\mathrm{d}z} - \frac{k}{\sigma^2}\delta p\right), \tag{A6.3}$$

$$\delta p \propto \rho c^2 \mathrm{e}^{-kz}\chi, \tag{A6.4}$$

where

$$\sigma^2 = \frac{\omega^2}{gk}. \tag{A6.5}$$

In view of the continuity of ρ, c^2 and δp, this implies that $\mathrm{d}\chi/\mathrm{d}z$ is continuous at $z = z_0$. Hence, the boundary condition to be applied to χ at the top of the polytropic region $z \geqslant z_0$ is

$$\frac{\mathrm{d}\chi}{\mathrm{d}z} + (\kappa - k)\chi = 0 \quad \text{at } z = z_0, \tag{A6.6}$$

which is equivalent to eq. (5.2.8).

The eigenvalue equation (A6.6) can be simplified for low-frequency modes, for which $z_0 k = (\mu + 1)Hk \ll 1$. In that case one can expand the confluent hypergeometric function U for small values of its argument (e.g. Abramowitz and Stegun (1964)):

$$\chi(z) = \frac{\pi\chi_0}{\sin\mu\pi}\left[A_1^{-1}\left(1 - \frac{2kz}{\mu+2} + \cdots\right) - A_2^{-1}(2kz)^{-\mu-1}\left(1 + \frac{a+\mu+1}{\mu}2kz + \cdots\right)\right], \tag{A6.7}$$

where

$$A_1 = \Gamma(-1 - \mu - 1)\Gamma(\mu + 2), \tag{A6.8}$$

$$A_2 = \Gamma(-a)\Gamma(-\mu). \tag{A6.9}$$

Expression (A6.1) can also be expanded to first order in Hk, yielding

$$\kappa = \tfrac{1}{2}H^{-1} - K \simeq [\gamma^{-1}(\sigma^2 - \sigma^{-2}) + \sigma^{-2}]k. \tag{A6.10}$$

With the help of the identity

$$\Gamma(-z)\Gamma(z+1)\sin\pi z = -\pi \tag{A6.11}$$

the negative signs in the arguments of the gamma functions in the definitions (A6.8) and (A6.9) can be removed; in transforming $\Gamma(-a)$ it should be noticed that, since z_0 is in the evanescent region, the atmospheric correction to the eigenfrequency is small, $a \simeq n - 1$ (which is the value when $z_0 = 0$), and consequently $\Gamma(-a) \simeq (-1)^n[(\alpha - n + 1)\Gamma(n)]^{-1}$. Then, substituting eq. (A6.7) into eq. (A6.6) and retaining only the leading terms in the resulting expansion, yields

$$a - n + 1 \simeq -\frac{[\gamma^{-1}(\sigma^2 - \sigma^{-2}) + 2\sigma^{-2}]\Gamma(a + \mu + 2)}{\Gamma(n)\Gamma(\mu+2)\Gamma(\mu+3)}(2kz_0)^{\mu+2}. \tag{A6.12}$$

(On first performing this analysis, Belvedere et al. (1983) omitted a factor $(\mu + 2)$ in the denominator on the right-hand side of eq. (A6.12). I am very grateful to B. Roberts for pointing this out to me.) One can now substitute for a using eq. (5.7.3), and set

$$\sigma^2 = (1 + \delta)s_n^2, \tag{A6.13}$$

where δ is small and s_n is the value of σ for the complete polytrope ($z_0 = 0$) which satisfies eq. (5.7.6). Expanding eq. (A6.12) and retaining only the dominant term then yields

$$\delta \simeq -S_n\frac{(\mu+1)^{\mu+2}\Gamma(\mu+n+1)}{\Gamma(n)\Gamma(\mu+2)\Gamma(\mu+3)}(2Hk)^{\mu+2}, \tag{A6.14}$$

where

$$S_n = \frac{s_n^2 + (2\gamma - 1)s_n^{-2}}{(\mu+1)s_n^2 - (\mu\gamma - \mu - 1)s_n^{-2}}. \tag{A6.15}$$

For high-frequency p modes s_n^4 is large, and one can simplify the expression even further. Writing $s_n^2 \simeq \omega^2/gk \simeq 2\gamma n/(\mu + 1)$, and using the asymptotic relation

$$\frac{\Gamma(n+\mu+1)}{\Gamma(n)} \sim n^{\mu+1} \quad \text{as } n \to \infty \tag{A6.16}$$

and the properties of the equilibrium model given in appendix I, yields

$$\delta \simeq -[n(\mu+1)\Gamma(\mu+2)\Gamma(\mu+3)]^{-1}[\tfrac{1}{2}(\mu+1)\omega/\omega_{c0}]^{2(\mu+2)}, \tag{A6.17}$$

where ω_{c0} is the value of ω_c in the isothermal atmosphere. Equations (A6.13) and (A6.17) are asymptotically equivalent to eq. (5.7.10).

It is worth remarking at this point that the wave equation in the atmosphere admits a solution in which the displacement is horizontal and independent of height.

The disturbance is simply a horizontally propagating pure sound wave, uninfluenced by gravity, and has frequency $\omega = kc$. It is sometimes called the Lamb wave. It can exist only when c is independent of height, and under no other circumstances. Strictly speaking, this solution cannot be matched onto the solution in the thermally stratified polytrope, and so a pure Lamb wave cannot exist in real stellar atmospheres. However, if an essentially isothermal atmosphere is bounded above by a high-temperature corona, waves with $kH \gg 1$ can be channelled in the atmosphere by refraction, and, as was pointed out in section 5.7.2, $\omega \sim kc$ as $k \to \infty$. The vertical component, ξ, of the displacement must in general be nonzero, and must have at least one node. The wave, therefore, is really a member of the ordinary p-mode class. It is discussed further in section 5.7.

Oscillations with $k = 0$ must be treated separately. Although radial modes extend right to the centre of the star, where the plane-parallel envelope model no longer applies, for the purpose of assessing the influence of the isothermal atmosphere it is sufficient to study the oscillations in detail only in the outer layers, where the model does apply. The deep interior can be replaced by a rigid boundary placed at a node of the eigenfunction, at a depth $z = Z$, which, of course, must be regarded as a function of ω; the result of the calculation is most useful when expressed in a form that does not depend explicitly on Z.

In the plane-parallel approximation eq. (4.1.5) for radial modes reduces to

$$\frac{d^2\xi}{dz^2} + \frac{\gamma g}{c^2}\frac{d\xi}{dz} + \frac{\omega^2}{c^2}\xi = 0, \tag{A6.18}$$

where ξ is the displacement. In the polytropic envelope ($z > z_0$) this equation can be rewritten as

$$\frac{d^2\xi}{dz^2} + \frac{\mu+1}{z}\frac{d\xi}{dz} + \frac{(\mu+1)\omega^2}{\gamma g z}\xi = 0, \tag{A6.19}$$

whose general solution is

$$\xi = \xi_0\left(\frac{z}{z_0}\right)^{-\mu/2}\left[J_\mu\left(\frac{z}{z_1}\right)^{1/2} + \epsilon Y_\mu\left(\frac{z}{z_1}\right)^{1/2}\right], \tag{A6.20}$$

where J_μ and Y_μ are Bessel functions of the first and second kind, ξ_0 and ϵ are constants and

$$z_1 = \frac{\gamma g}{4(\mu+1)\omega^2}\,. \tag{A6.21}$$

In the isothermal atmosphere eq. (A6.18) becomes

$$\frac{d^2\xi}{dz^2} + \frac{1}{H}\frac{d\xi}{dz} + \frac{\omega^2}{c^2}\xi = 0, \tag{A6.22}$$

whose causal solution is

$$\xi = \xi_0[J_\mu(\lambda) + \epsilon Y_\mu(\lambda)]\,e^{-\kappa(z-z_0)}, \tag{A6.23}$$

where

$$\kappa = \frac{1}{2H}\left[1 - \left(1 - \frac{\omega^2}{\omega_{c0}^2}\right)^{1/2}\right], \tag{A6.24}$$

$$\lambda = \left(\frac{z_0}{z_1}\right)^{1/2}. \tag{A6.25}$$

Continuity of ξ has been ensured by the choice of the normalization constant in eq. (A6.23). From eqs. (3.11) and (4.1.3) follows that continuity of δp then requires continuity of $d\delta\xi/dz$. This is expressed as

$$J'_\mu(\lambda) + \epsilon Y'_\mu(\lambda) = \sqrt{z_0 z_1}(\mu z_0^{-1} - 2\kappa)[J_\mu(\lambda) + \epsilon Y_\mu(\lambda)], \tag{A6.26}$$

in which the primes denote differentiation with respect to the argument. When $\omega \ll \omega_{c0}$, then $\lambda \ll 1$, and one can expand the Bessel functions in power series in λ (see, e.g., Abramowitz and Stegun (1964)), yielding

$$\epsilon \simeq \frac{\pi}{\Gamma(\mu+2)\Gamma(\mu+3)}\left(\frac{1}{2}\lambda\right)^{2(\mu+2)}. \tag{A6.27}$$

Formally, the eigenvalue equation is now obtained by the condition $\xi = 0$ at the undetermined depth $z = Z$. For $n \gg 1$, this can be simplified by expanding the Bessel functions in eq. (A6.20) for large arguments, yielding

$$\omega \simeq (n + \tfrac{1}{2}\mu - \tfrac{1}{4})(1+\delta)\omega_0, \tag{A6.28}$$

where ω_0 satisfies

$$\omega_0^2 = \frac{\pi^2\gamma g}{4(\mu+1)Z}$$

and is now to be considered to be analogous to the characteristic frequency defined by eq. (4.8.46), which removes its explicit dependence on Z, and

$$\delta \simeq \frac{\epsilon}{n\pi} \simeq \frac{1}{n\Gamma(\mu+2)\Gamma(\mu+3)}\left[\frac{1}{2}(\mu+1)\frac{\omega}{\omega_{c0}}\right]^{2(\mu+2)} \tag{A6.29}$$

$$\simeq \frac{1}{\Gamma(\mu+2)\Gamma(\mu+3)}\left[\frac{1}{2}(\mu+1)\omega_{c0}^{-1}\right]^{2(\mu+2)}\omega_0\omega^{2\mu+3}. \tag{A6.30}$$

Equation (A6.28), with $\delta = 0$, is similar to the eigenvalue equation (4.8.45) obtained from the JWKB approximation.

Appendix 7. Acoustic oscillations of an isothermal sphere

Acoustic oscillations of a homogeneous isothermal sphere with radius R in the absence of gravity have been discussed by Rayleigh (1896). The Lagrangian pressure perturbation δp satisfies

$$\nabla^2 \delta p - \frac{1}{c^2}\frac{\partial^2 \delta p}{\partial t^2} = 0, \tag{A7.1}$$

which can be separated in the usual way:

$$\delta p(r,t) = \mathrm{Re}[r^{-1}\Psi(r)P_l^m(\cos\theta)\,\mathrm{e}^{im\phi - i\omega t}], \tag{A7.2}$$

whence the radial amplitude function satisfies

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \left(\frac{\omega^2}{c^2} - \frac{L^2}{r^2}\right)\Psi = 0, \tag{A7.3}$$

where $L^2 = l(l+1)$. Equation (A7.3) is simply eq. (5.4.7), appropriately simplified, aside from a scaling factor in the definition of Ψ. Its regular solution is

$$\Psi = r^{1/2}J_\lambda\left(\frac{\omega r}{c}\right), \tag{A7.4}$$

where J_λ is the Bessel function of the first kind and $\lambda = l + \frac{1}{2}$. If the perturbed surface of the sphere is maintained at a constant pressure (which could be imagined to be accomplished by imbedding the sphere in a very hot diffuse gas, whose inertia is negligible), then

$$\delta p = 0 \quad \text{at } r = R, \tag{A7.5}$$

from which follows the eigenvalue equation

$$\omega = \pi^{-1} j_{n,\lambda}\,\omega_0, \tag{A7.6}$$

where

$$\omega_0 = \pi c/R \tag{A7.7}$$

and $j_{n,\lambda}$ is the nth zero of J_λ.

An expansion of the zeros of J_λ, valid for high n/l, is

$$j_{n,\lambda} = (n + \tfrac{1}{2}l)\pi - \frac{l(l+l)}{2\pi(n+l/2)} + \cdots, \tag{A7.8}$$

from which follows that

$$\omega \simeq (n + \tfrac{1}{2}l)\omega_0 - L^2\omega_0^2/2\pi^2\omega + \cdots, \tag{A7.9}$$

when $n/l \gg 1$. The leading term yields eq. (5.8.15). Equation (A7.7) is a special case of eq. (5.8.30), and it is evident that eq. (A7.9) is a special case of eq. (5.8.31).

Appendix 8. The scaling factor in the Abel sound-speed inversion

Let the sound speed, c, be approximated by

$$c = c_0\left(\frac{z}{z_0}\right)^\beta, \tag{A8.1}$$

where β is a constant, z is the depth below the stellar surface $r = R$ and $c_0 = c(z_0)$, z_0 being the shallowest lower turning point, which I take to be associated with a mode with order n_0 and degree l_0. I will assume that the data contain high-degree modes, so that $z_0/R \ll 1$. Provided interest is in values of $R - r$ much greater than z_0, then $a \gg w_0 \geqslant a'$ in eq. (6.10), and

$$\Lambda \simeq -\ln\left(1 - \frac{z_0}{R}\right) - \frac{2}{\pi R}\int_0^{z_0} \sin^{-1}\left(\frac{z}{z_0}\right)^\beta \mathrm{d}z \tag{A8.2}$$

$$\simeq \left\{1 - \frac{2}{\pi}\left[\left(\frac{\pi}{2}\right)^{1/\beta} - \int_0^1 \frac{u^{1/\beta}\,\mathrm{d}u}{\sqrt{1-u^2}}\right]\right\}\frac{z_0}{R} \tag{A8.3}$$

$$= \left[1 - \left(\frac{\pi}{2}\right)^{2(b-1)} + \frac{\Gamma(b)}{\sqrt{\pi}\,\Gamma(b+\frac{1}{2})}\right]\frac{z_0}{R}, \tag{A8.4}$$

where Γ is the gamma function and $b = (1+\beta)/2\beta$. Equation (A8.3) can be obtained from eq. (A8.2) with the help of the substitution $u = (z/z_0)^\beta$.

The surface layers of the Sun are roughly polytropic, so we expect $\beta \simeq 0.5$. In that case, eq. (A8.4) can be approximated by

$$\Lambda \simeq (1 - 0.37\beta^{-1.54})\frac{z_0}{R}, \tag{A8.5}$$

the coefficient of z_0/R being in error by no more than about 0.02 for $0.4 \lesssim \beta \lesssim 0.8$. One can estimate z_0 from the turning-point condition $\omega = kc_0 \simeq l_0 R^{-1} c_0$, coupled with the polytropic approximation to the eigenfrequency. Using just the leading term of eq. (5.7.7) (note, in passing, that the β in eq. (5.7.7) is not related to the β here), one obtains

$$\frac{z_0}{R} \simeq 2(n_0 + \tfrac{1}{2}\mu) l_0^{-1}. \tag{A8.6}$$

Taking the polytropic value 0.5 for β then yields

$$\Lambda \simeq -0.14(n_0 + \tfrac{1}{2}\mu) l_0^{-1}. \tag{A8.7}$$

Equation (A8.7) actually underestimates the magnitude of Λ, since the coefficient in eq. (A8.5) changes sign at $\beta \simeq 0.52$, close to the polytropic value that was adopted. It is therefore prudent to use a more realistic value of β. From eq. (6.1), using eq. (A8.1) for c, one obtains

$$\mathcal{F}(w) = \frac{\pi z_0}{A c_0} \left(\frac{Rw}{c_0}\right)^{2(b-1)}, \tag{A8.8}$$

where

$$A = 2\sqrt{\pi}\,\frac{\Gamma(b-1/2)}{\Gamma(b-1)}. \tag{A8.9}$$

Note that eqs. (A8.8) and (6.1) imply that

$$\frac{z_0}{R} = A(n_0 + \alpha) l_0^{-1}, \tag{A8.10}$$

since $R w_0 / c_0 = 1$.

Current solar observations satisfy $\pi(n+\alpha)/\omega \propto w^{0.65}$ with $\alpha \simeq 1.5$ when $l \simeq 10^3$, suggesting that in the vicinity of the lower turning points of those modes the sound speed is more accurately represented by eq. (A8.1) with $\beta \simeq 0.6$. With this value, eqs. (A8.4), (A8.9) and (A8.10) then provide the improved estimate: $A = 1.42$ and

$$\Lambda \simeq 0.28(n_0 + 1.5) l_0^{-1}. \tag{A8.11}$$

Appendix 9. Invariants from eikonal equations under symmetries in a sphere

Consider the local dispersion relation, such as eq. (8.3.2), to be written with respect to spherical polar coordinates:

$$\omega = W(k_r, k_\theta, k_\phi, r, \theta, \phi, t). \tag{A9.1}$$

The transformation of the wave number $\boldsymbol{k}$ from Cartesian components $(k_x, k_y, k_z) \equiv (k_1, k_2, k_3)$ to spherical polar components (k_r, k_θ, k_ϕ) is given by

$$k_r = (k_x \cos\phi + k_y \sin\phi)\sin\theta + k_z \cos\theta, \tag{A9.2}$$

$$k_\theta = (k_x \cos\phi + k_y \sin\phi)\cos\theta - k_z \sin\theta, \tag{A9.3}$$

$$k_\phi = -k_x \sin\phi + k_y \cos\phi. \tag{A9.4}$$

The inverse transformation is

$$k_x = (k_r \sin\theta + k_\theta \cos\theta)\cos\phi - k_\theta \sin\phi, \tag{A9.5}$$

$$k_y = (k_r \sin\theta + k_\theta \cos\theta)\sin\phi + k_\phi \cos\phi, \tag{A9.6}$$

$$k_z = k_r \cos\theta - k_\theta \sin\theta. \tag{A9.7}$$

Any other vector, such as the group velocity $\boldsymbol{v}$, transforms similarly.

The Cartesian components of the eikonal equation (8.3.6) can be written as

$$\frac{\partial k_i}{\partial t} + v_r \frac{\partial k_i}{\partial r} + \frac{v_\theta}{r}\frac{\partial k_i}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial k_i}{\partial \phi} = -\left(\frac{\partial W}{\partial x_i}\right)_{k_j}, \quad i = 1,2,3, \tag{A9.8}$$

where $(x_1, x_2, x_3) \equiv (x, y, z)$ and the partial derivatives $(\partial W/\partial x_i)_{k_j}$ are taken at constant Cartesian components k_j of $\boldsymbol{k}$, and, of course, at constant Cartesian coordinates x_l $(l \neq i)$. The quantities (v_r, v_θ, v_ϕ) are the spherical polar components of the group velocity, $\boldsymbol{v}$, defined by eq. (8.3.5), which can easily be shown to be given by $(\partial W/\partial k_r, \partial W/\partial k_\theta, \partial W/\partial k_\phi)$.

To construct the eikonal equation for k_ϕ, multiply eq. (A9.8) for $i = 2$ by $\cos\phi$, for $i = 1$ by $\sin\phi$, and subtract. With the help of transformations (A9.4)–(A9.6), the result can be written as

$$\frac{\partial k_\phi}{\partial t} + \boldsymbol{v}\cdot\nabla k_\phi + \frac{v_\phi k_r}{r} + \frac{v_\phi k_\theta \cot\theta}{r} = -\frac{1}{r\sin\theta}\left(\frac{\partial W}{\partial \phi}\right)_{k_i}. \tag{A9.9}$$

The partial derivative of W with respect to ϕ is now taken at constant r and ϕ, but the parentheses with subscript k_i have been retained as a reminder that it is still at constant Cartesian components of $\boldsymbol{k}$.

The equation for k_θ is obtained by multiplying eq. (A9.8) for $i = 1$ by $\cos\theta\cos\phi$, for $i = 2$ by $\cos\theta\sin\phi$ and adding, and then subtracting $\sin\theta$ times the equation for $i = 3$, yielding

$$\frac{\partial k_\theta}{\partial t} + \boldsymbol{v}\cdot\nabla k_\theta + \frac{v_\theta k_r}{r} - \frac{v_\phi k_\phi \cot\theta}{r} = -\frac{1}{r}\left(\frac{\partial W}{\partial \theta}\right)_{k_i}. \tag{A9.10}$$

It is now necessary to transform the derivatives of W. The ϕ derivative, e.g., can be expanded thus:

$$\left(\frac{\partial W}{\partial \phi}\right)_{k_i} = \frac{\partial W}{\partial \phi} + \frac{\partial W}{\partial k_r}\left(\frac{\partial k_r}{\partial \phi}\right)_{k_i} + \frac{\partial W}{\partial k_\theta}\left(\frac{\partial k_\theta}{\partial \phi}\right)_{k_i} + \frac{\partial W}{\partial k_\phi}\left(\frac{\partial k_\phi}{\partial \phi}\right)_{k_i}, \tag{A9.11}$$

where the partial derivatives of W are now taken at constant values of its other arguments listed in eq. (A9.1). The partial derivatives of (k_r, k_θ, k_ϕ) are obtained by differentiating eqs. (A9.2)–(A9.4). The result is

$$\begin{aligned}\left(\frac{\partial W}{\partial \phi}\right)_{k_i} &= \frac{\partial W}{\partial \phi} + (v_r k_\phi - v_\phi k_r)\sin\theta + (v_\theta k_\phi - v_\phi k_\theta)\cos\theta \\ &= \frac{\partial W}{\partial \phi} + (\boldsymbol{v} \times \boldsymbol{k})_z .\end{aligned} \tag{A9.12}$$

Similarly, one obtains

$$\left(\frac{\partial W}{\partial \theta}\right)_{k_i} = \frac{\partial W}{\partial \theta} + v_r k_\theta - v_\theta k_r = \frac{\partial W}{\partial \theta} + (\boldsymbol{v} \times \boldsymbol{k})_\phi . \tag{A9.13}$$

Equations (A9.9) and (A9.12) can now be combined to give

$$\frac{\mathrm{d}}{\mathrm{d}t}(r k_\phi \sin\theta) = -\frac{\partial W}{\partial \phi}, \tag{A9.14}$$

where the full time derivative, $\mathrm{d}/\mathrm{d}t$, along the ray path is defined by eq. (8.3.4). Equations (A9.9) and (A9.10) can be combined to provide an equation for $k_\mathrm{h}^2 = k_\theta^2 + k_\phi^2$, which, with the help of eqs. (A9.12) and (A9.13) becomes

$$\begin{aligned}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}t}(r^2 k_\mathrm{h}^2) &= -\frac{2}{r}[(\boldsymbol{v} \times \boldsymbol{k}) \times \boldsymbol{k}]_r - 2\boldsymbol{k}_\mathrm{h} \cdot (\nabla W)_{k_i} \\ &= -\frac{2}{r}(\boldsymbol{v} \times \boldsymbol{k})_r k_\phi \cot\theta - 2\boldsymbol{k}_\mathrm{h} \cdot \nabla W,\end{aligned} \tag{A9.15}$$

where $\boldsymbol{k}_\mathrm{h} = (0, k_\theta, k_\phi)$ is the horizontal component of $\boldsymbol{k}$, the gradient $(\nabla W)_{k_i}$ is taken at constant Cartesian components of $\boldsymbol{k}$ and the gradient ∇W is taken at constant spherical polar components of $\boldsymbol{k}$. The second part of eq. (A9.15) is obtained from the first simply by expanding the gradient, regarding W as a function of the spherical polar components of $\boldsymbol{k}$ and $\boldsymbol{x}$, as in eq. (A9.1), and using the transformation (A9.2)–(A9.4) to evaluate the spatial derivatives of the components k_r, k_θ, k_ϕ of $\boldsymbol{k}$ at constant Cartesian components k_i, and then regrouping the terms.

The invariants (8.3.10) and (8.3.11) follow immediately. If W is not explicitly dependent on ϕ, then $\partial W/\partial\phi = 0$ and, according to eq. (A9.14), $r\sin\theta k_\phi$ is constant along a ray path. If the background state is genuinely spherically symmetrical, so that $W = W(k_r, k_\mathrm{h}, r, t)$, then $\boldsymbol{k}_\mathrm{h}\cdot\nabla W = 0$ and $\boldsymbol{v}\times\boldsymbol{k} = (\partial W/\partial k_\mathrm{h})k^{-1}(\boldsymbol{k}_\mathrm{h}\times\boldsymbol{k}_r)$, where $\boldsymbol{k}_r = (k_r, 0, 0)$, from which follows that $(\boldsymbol{v}\times\boldsymbol{k})_r = 0$; hence rk_h is also invariant along a ray path. Notice that the latter condition requires genuine spherical symmetry. If the stratification of the background state were spherically symmetrical, but the entire system were pervaded by, say, a uniform magnetic field, which defines a preferred direction, then W would depend explicitly not only on k_h but also on k_ϕ, and the conditions for the invariance of rk_h would no longer be satisfied.

Appendix 10. Asymptotic associated Legendre functions

The associated Legendre function of the first kind P_l^m with degree l and order $m \geqslant 0$ is the regular solution of the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P_l^m = 0. \tag{A10.1}$$

This can be reduced to standard form by the substitution

$$P_l^m = \csc^{1/2}\theta\, Q. \tag{A10.2}$$

Then Q satisfies

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\theta^2} + (L^2 - \tilde{m}^2\csc^2\theta)Q = 0, \tag{A10.3}$$

where

$$L = l + \tfrac{1}{2} \tag{A10.4}$$

and

$$\tilde{m}^2 = m^2 - \tfrac{1}{4}. \tag{A10.5}$$

Equation (A10.3) has turning points at $\theta = \theta_1 := \cos^{-1}\mu_\mathrm{t}$ and $\theta = \theta_2 := \cos^{-1}(-\mu_\mathrm{t}) = \pi - \theta_1$, where

$$\mu_\mathrm{t}^2 = 1 - \tilde{m}^2/L^2.$$

Asymptotic solutions can be obtained by Olver's method, which is summarized in section 4.4.

To accommodate both turning points, two separate approximations in terms of Airy functions are required, which are then matched in their common region of validity: the region at low latitudes far from the turning points, where $L^2 - \tilde{m}^2 \csc^2\theta \gg 1$ and the solution is oscillatory. In view of the symmetry of the problem, both turning points are treated similarly, the outcome being a solution which is either symmetric or antisymmetric about the equator. An appropriate new independent variable is obtained from the Liouville transformation:

$$\psi = \pm\,\mathrm{sgn}(\theta - \theta_t)\left(\tfrac{3}{2}\int_{\theta_t}^{\theta} \left|L^2 - \tilde{m}^2\csc^2\theta\right|^{1/2} \mathrm{d}\theta\right)^{2/3}, \qquad \text{(A10.6)}$$

the plus sign and $\theta_t = \theta_1$ being employed for the representation where θ is less than and not very close to θ_2, which I call region 1, the minus sign and $\theta_t = \theta_2$ being employed in region 2, where θ is greater than and not very close to θ_1. With respect to this variable, the asymptotic representation with $L \gg 1$ of the solution of eq. (A10.3) that in the appropriate evanescent region decays away from the turning point, is

$$Q \sim \sqrt{2}\,Q_i \left|(L^2 - \tilde{m}^2\csc^2\theta)^{-1}\psi\right|^{1/4} \mathrm{Ai}(-\psi), \qquad \text{(A10.7)}$$

where i labels the region in which the representation is valid and the Q_i are constants, which may possibly take different values for the two representations.

Several different techniques are available for evaluating the integral in the definition (A10.6) of ψ. Alternatively, one can verify the following expressions by differentiation, confirming that they vanish at $\theta = \theta_t$:

$$\psi = \left\{\frac{3}{2}\left[L\cos^{-1}\left(\pm\frac{\cos\theta}{\mu_t}\right) - \tilde{m}\cos^{-1}\left(\pm\frac{\tilde{m}\cot\theta}{L\mu_t}\right)\right]\right\}^{2/3}, \qquad |\cos\theta| < \mu_t \qquad \text{(A10.8)}$$

$$\psi = -\left\{\frac{3}{2}\left[L\ln\left(\frac{\pm L\cos\theta + (\tilde{m}^2 - L^2\sin^2\theta)^{1/2}}{(L^2 - \tilde{m}^2)^{1/2}}\right) \mp \frac{1}{2}\tilde{m}\ln\left(\frac{(\tilde{m}^2 - L^2\sin^2\theta)^{1/2} + \tilde{m}\cos\theta}{(\tilde{m}^2 - L^2\sin^2\theta)^{1/2} - \tilde{m}\cos\theta}\right)\right]\right\}^{2/3}, \qquad |\cos\theta| > \mu_t. \qquad \text{(A10.9)}$$

For the representation in region 1 the plus signs are required and for the representation in region 2, which can be obtained from the first representation by replacing θ by $\pi - \theta$, the minus signs are required.

The solution in the oscillatory region can be obtained by replacing the Airy function, Ai, in eq. (A10.7) by the leading term of its asymptotic expansion (4.4.25). Together with eq. (A10.2), this yields

$$P_l^m \sim \left(\frac{2}{\pi}\right)^{1/2} Q_i(L^2\sin^2\theta - \tilde{m}^2)^{-1/4} \times \sin\left[L\cos^{-1}\left(\pm\frac{\cos\theta}{\mu_t}\right) - \tilde{m}\cos^{-1}\left(\pm\frac{\tilde{m}\cot\theta}{L\mu_t}\right) + \frac{\pi}{4}\right]. \qquad \text{(A10.10)}$$

These two representations must be asymptotically equivalent, which can be so only if $\tilde{m}$ is an integer. This suggests replacing eq. (A10.5) by $\tilde{m} = m$ (although at this stage of the argument we must admit the possibility that $\tilde{m} = m + k$, where k is an integer that is not zero). It then follows that when $l - m$ is an even integer, $Q_2 = Q_1$ and the solution is an even function of $\cos\theta$. When $l - m$ is odd, $Q_2 = -Q_1$ and P_l^m is an odd function of $\cos\theta$.

It is instructive now to consider the behaviour of the solution in the neighbourhood of the singular points, $\theta = 0$ and $\theta = \pi$. Expanding the regular solution of eq. (A10.1) about $\theta = 0$ yields $P_l^m \propto \theta^m$. Introducing the asymptotic approximation (4.4.26) into the solution (A10.7), (A10.2) yields

$$P_l^m \sim (2\pi\tilde{m})^{-1/2} Q_1 \theta^{\tilde{m}} \quad \text{as } \theta \to 0; \qquad \text{(A10.11)}$$

there is a similar expression in the limit $\theta \to \pi$. This approximation can therefore be made exact when $\tilde{m}$ is once again replaced by m, confirming the suggestion from the matching in the oscillatory region. Notice also that when $\tilde{m} = m$, the representation (A10.10) in the oscillatory region is proportional to the function $\mathcal{P}_l^m$ defined by eq. (8.6.16).

It should be noticed that the matching of the two representations is possible only if the oscillatory region exists. Therefore $L^2\sin^2\theta$ must exceed m^2 somewhere, which implies $|m| < L$ and hence $|m| \leqslant l$. It is also evident that P_l^m has $l - m$ zeros in $0 < \theta < \pi$.

It remains to determine the constant Q_1. Adopting the normalization (e.g. Abramowitz and Stegun (1964)):

$$I_{lm} := \int_{-1}^{1} [P_l^m(\mu)]^2\,\mathrm{d}\mu = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}, \qquad \text{(A10.12)}$$

one obtains from the asymptotic expression (A10.10)

$$I_{lm} \sim \pi^{-1} Q_1^2 L^{-1} \int_{-\mu_1}^{\mu_1} (\mu_1^2 - \mu^2)^{-1/2} \,\mathrm{d}\mu = L^{-1} Q_1^2, \tag{A10.13}$$

from which one deduces that

$$Q_1 = \left(\frac{(l+m)!}{(l-m)!} \right)^{1/2}. \tag{A10.14}$$

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