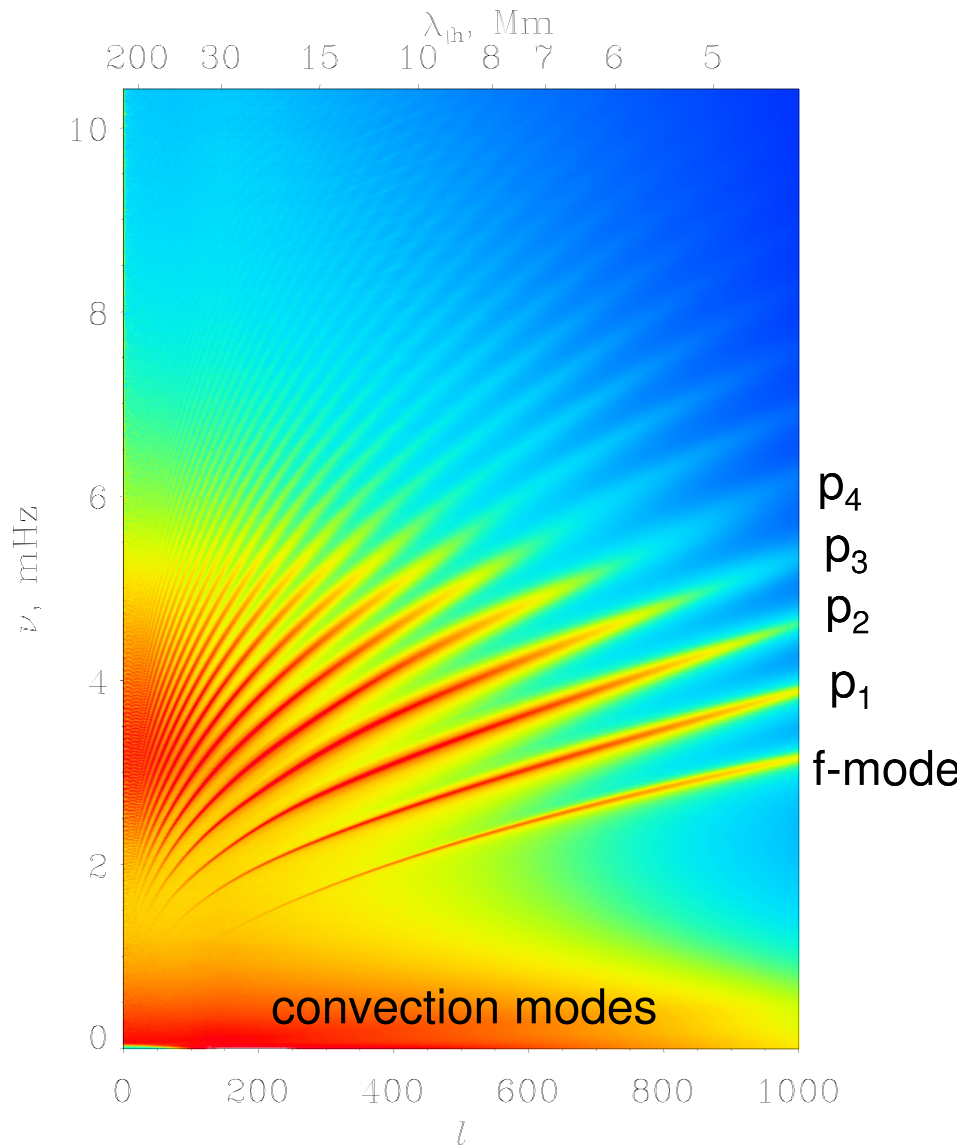
10. Principles of helioseismology: Global helioseismology

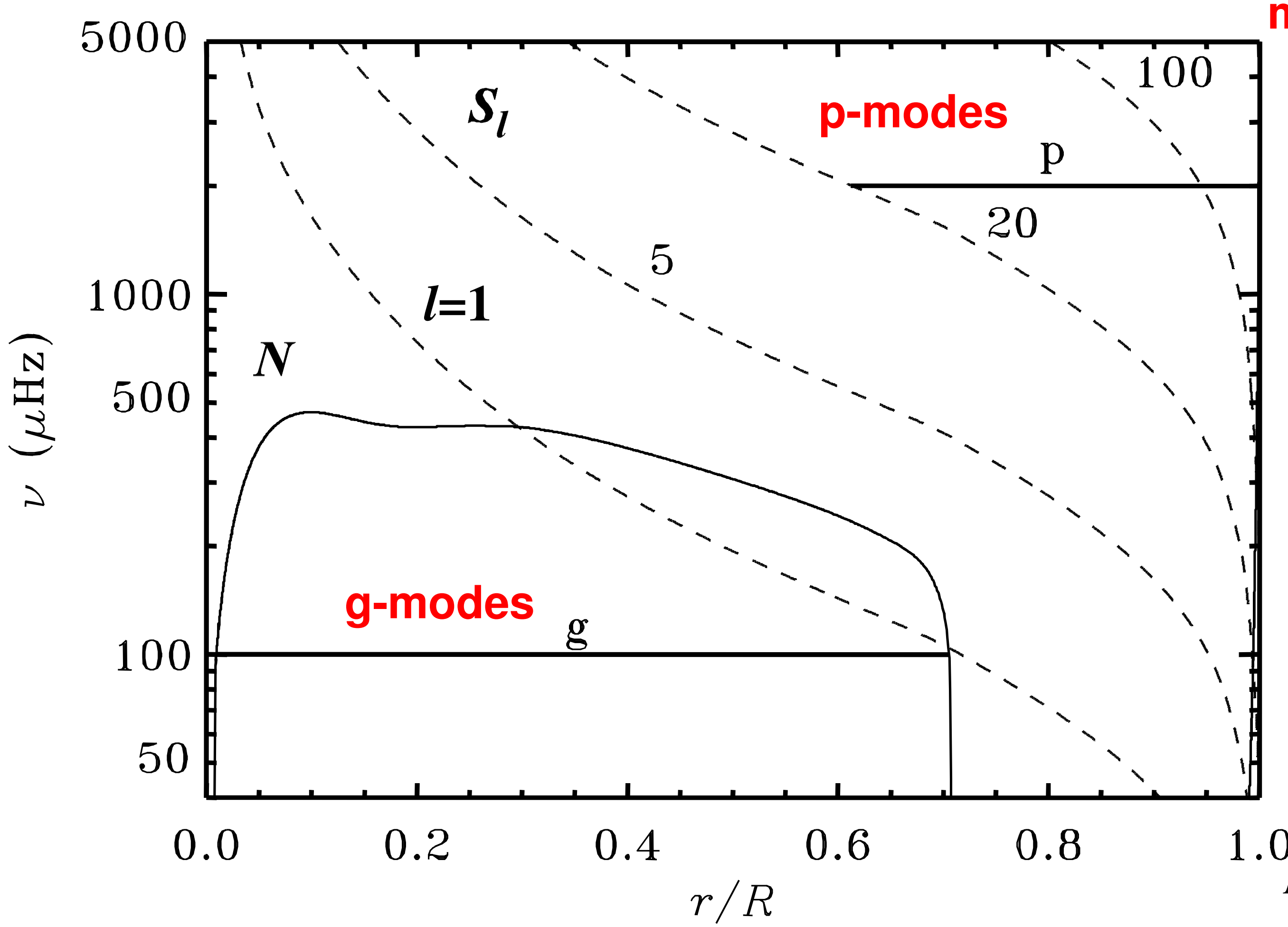
Oscillation power spectrum

- Spherical harmonic transform – oscillation signal is represented in terms of spherical harmonics of angular degree l.

Only p-modes have been observed. Global helioseismology is based on inferences of the interior structure and rotation from the p-mode frequencies. The frequencies are measured by fitting the peaks in the power spectrum.



Propagation diagram of solar oscillations



p-modes (acoustic modes):

$$\omega > S_l \quad \omega > N$$

g-modes (internal gravity modes):

$$\omega < S_l \quad \omega < N$$

the Lamb frequency:

$$S_l^2 = \frac{L^2 c^2}{r^2}$$

$$L^2 = l(l+1)$$

the Brunt-Vaisala frequency:

$$N^2 = g\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right)$$

Buoyancy (Brunt-Vaisala) frequency N **, and Lamb frequency** S_l **for** $l = 1, 5, 20$ **and 100 vs. fractional radius** r/R **for a standard solar model. The horizontal lines indicate the trapping regions for a g mode with frequency** $\nu = 100\mu$ **Hz, and for a p mode of degree** $l = 20$ **and** $\nu = 2000\mu$ **Hz.**

Properties of Solar Normal Modes

Equation $$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} + \frac{S_l^2}{c^2\omega^2}\left(N^2 - \omega^2\right)$$

represents a **dispersion relation of solar waves**.

It relates frequency ω with radial wavenumber k_r and angular order l.

If $\omega^2 >> N^2$ then

$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{S_l^2}{c^2}$$

or $$\omega^2 = \omega_c^2 + k_r^2 c^2 + k_h^2 c^2,$$

where $k_h = \frac{S_l}{c} = \frac{L}{r} = \frac{\sqrt{l(l+1)}}{r}$ is **the horizontal wave number**.

Then, $k^2 = k_r^2 + k_h^2$ is the squared total wavenumber.

Finally, $$\omega^2 = \omega_c^2 + k^2 c^2.$$

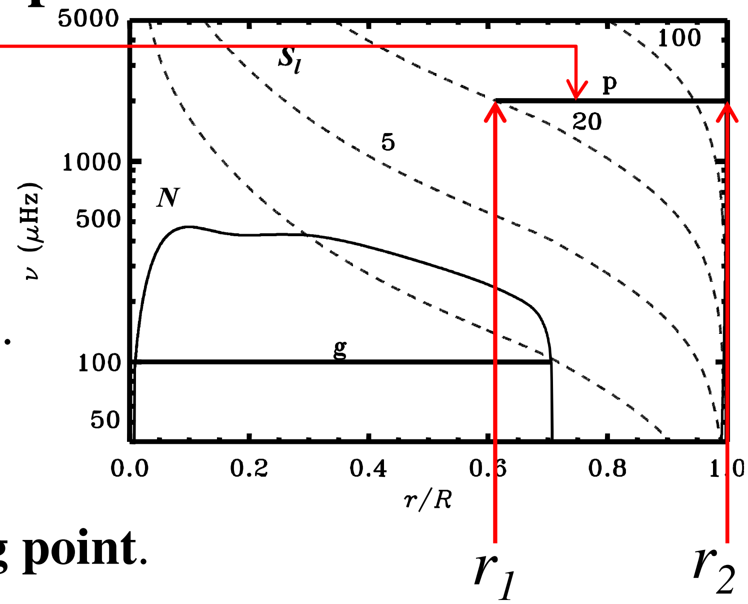
This is **the dispersion relation for acoustic (p) modes**;

$\omega_c = \frac{c}{2H}$ **is the acoustic cutoff frequency;** $\nu_c = \omega_c / 2\pi \approx 5$ mHz

Estimate frequencies of normal modes for p-modes:

their propagating region: $k_r^2 > 0$

the turning points $k_r^2 = 0$: $$\omega^2 = \omega_c^2 + \frac{L^2 c^2}{r^2}$$



For the lower turning point in the interior: $\omega_c \ll \omega$.

Then, $\omega \approx \dfrac{Lc}{r}$, or

$$\frac{c(r_1)}{r_1} = \frac{\omega}{L}$$ is the equation for **the lower turning point**.

The upper turning point: $\omega_c(r_2) \approx \omega$.

$\omega_c(r)$ is a steep function of r near the surface, thus $r_2 \approx R$.

The resonant condition for p-modes is: $$\int_{r_1}^{R} \sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}\, dr = \pi(n+\alpha)$$

Abel integral equation for the sound-speed profile c(r).

Duvall's law (asymptotic p-mode relation)

Consider the p-mode dispersion relation:

$$\int_{r_1}^{R} k_r dr = \pi(n+\alpha)$$

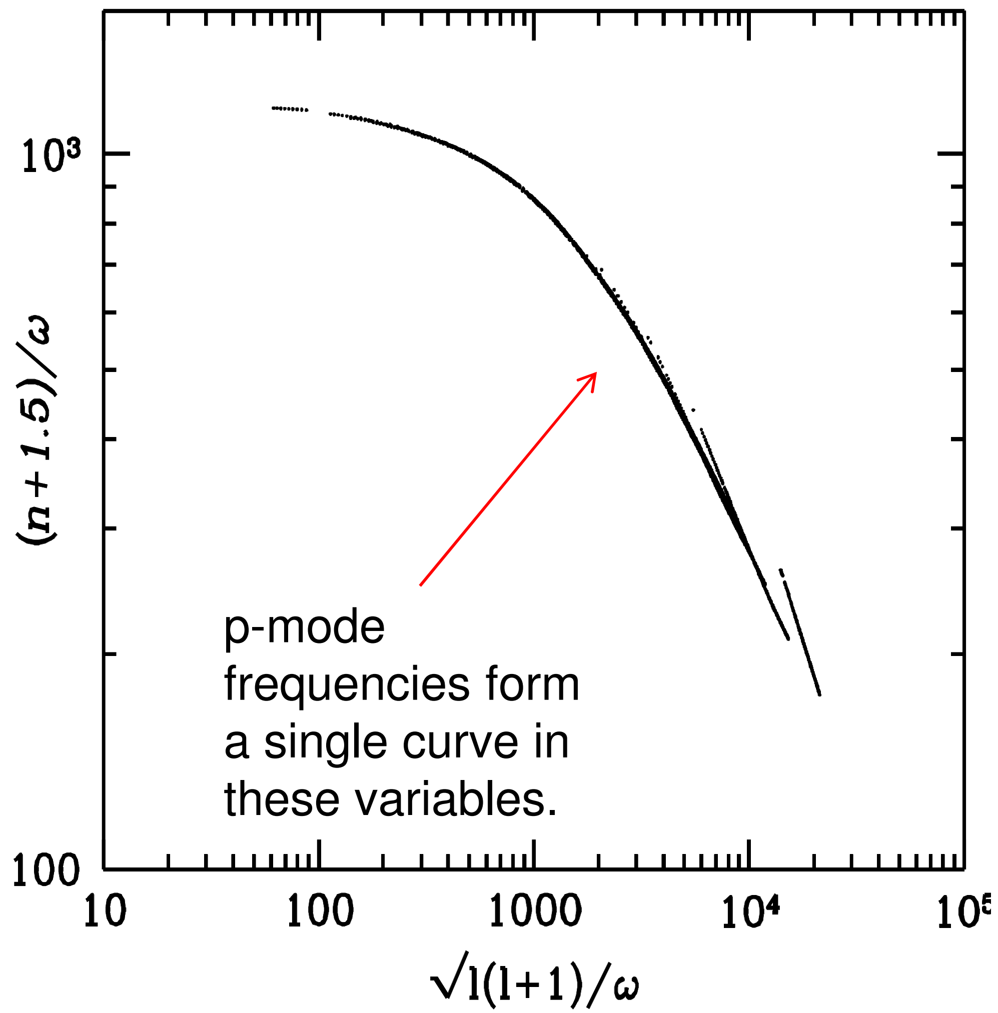
$$\int_{r_1}^{R} \left(\frac{\omega^2}{c^2} - \frac{L^2}{r^2}\right)^{1/2} dr = \pi(n+\alpha)$$

Dividing left and right-hand sides by ω we get:

$$\int_{r_1}^{R} \left(\frac{r^2}{c^2} - \frac{L^2}{\omega^2}\right)^{1/2} \frac{dr}{r} = \frac{\pi(n+\alpha)}{\omega}$$

Radius r_1 (or r_t) of the lower turning point depends only on ratio L/ω. Hence, the left-hand side is a function of L/ω:

$$F\left(\frac{L}{\omega}\right) = \frac{\pi(n+\alpha)}{\omega}$$

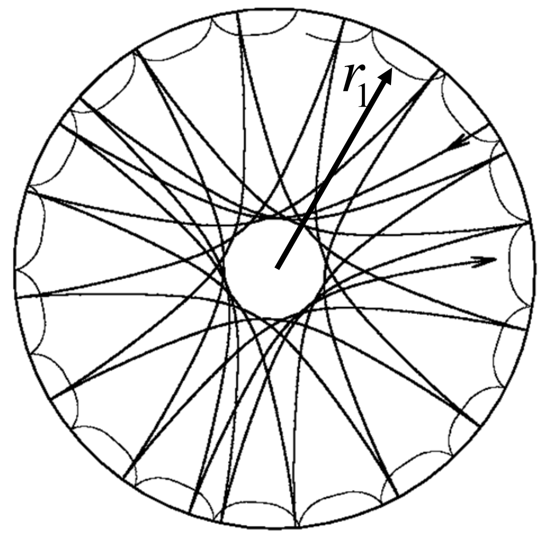


where $L = \sqrt{l(l+1)}$ $\alpha \approx 1.5$

Asymptotic sound-speed inversion

To find corrections to the standard solar model we consider small perturbations to the sound speed profile and oscillation frequencies, and linearize the dispersion relation:

$$\int_{r_t}^{R}\left[\frac{(\omega+\Delta\omega)^2}{(c+\Delta c)^2}-\frac{L^2}{r^2}\right]^{1/2} dr = \pi(n+\alpha).$$



$$\frac{\Delta\omega}{\omega}\int_{r_t}^{R}\frac{dr}{c\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{1/2}}=\int_{r_t}^{R}\frac{\Delta c}{c}\frac{dr}{c\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{1/2}}.$$

$$\frac{\Delta\omega}{\omega}T=\int_{r_t}^{R}\frac{\Delta c}{c}\frac{dr}{c\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{1/2}},$$ where $$T=\int_{r_t}^{R}\frac{dr}{c\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{1/2}},$$

This equation has a simple physical interpretation: T is the travel time of acoustic waves to travel along the p-mode ray path; the right-hand side integral is an average of the sound-speed perturbations along this ray path.

This is a linear integral equation with a singular point at $r=r_t$

Asymptotic sound-speed inversion

$$\frac{\Delta\omega}{\omega} = \frac{1}{T}\int_{r_t}^{R} \frac{\Delta c}{c} \frac{dr}{c\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{1/2}},$$

Here c(r) is the sound-speed profile of the standard solar model, and $\omega(l,n)$ are the p-mode frequencies calculated for the standard solar model. This equation can be reduced to the standard Abel integral equation by making a substitution of variables. The new variables are:

$x = \frac{\omega^2}{L^2}$ and $y = \frac{c^2}{r^2}$, where x is a measured quantity, y is unknown function

x can be considered as a continuous function according to the Duvall's law

$$T\frac{\Delta\omega}{\omega} = \int_{r_t}^{R} \frac{\Delta c}{c} \frac{x^{1/2}dr}{c(x-y)^{1/2}}, \quad dr = \frac{dy}{dy/dr} = \frac{r}{y}\frac{dy}{\frac{d\log y}{d\log r}}$$

$$\log y = 2\log c - 2\log r, \quad \frac{d\log y}{d\log r} = 2\left(\frac{d\log c}{d\log r} - 1\right)$$

Asymptotic sound-speed inversion

$$T\frac{\Delta\omega}{\omega}\frac{1}{x^{1/2}} = \int_{y_S}^{x}\frac{\Delta c}{c}\frac{1}{2y^{3/2}\left(1-\frac{d\log c}{d\log r}\right)}\frac{dy}{(x-y)^{1/2}},$$

$$x = \frac{\omega^2}{L^2} \quad \text{and} \quad y = \frac{c^2}{r^2}$$

$y_S = \frac{c(R)}{R}$ - the surface value at $r = R$, $y_S \approx 0$

$$F(x) = \int_0^x \frac{f(y)dy}{\sqrt{x-y}},$$

This is the Abel integral equation.

where $F(x) = T\frac{\Delta\omega}{\omega}\frac{1}{\sqrt{x}}$, $f(y) = \frac{\Delta c}{c}\frac{1}{2y^{3/2}\left(1-\frac{d\log c}{d\log r}\right)}$.

The solution to the Abel integral equation is:

$$f(y) = \frac{1}{\pi}\frac{d}{dz}\int_0^z \frac{F(x)dx}{\sqrt{z-x}}.$$

It gives $\Delta c/c(r)$.

Helioseismic Inverse Problem

In the asymptotic (high-frequency of short wavelength) approximation the oscillation frequencies depend only on the sound-speed profile. This dependence is expressed in terms of the Abel integral equation that can be solved analytically.

In the general case, the relation between the frequencies and internal properties is non-linear, and there is no analytical solution. **Generally, the frequencies determined from the oscillation equation depend on the density, $\rho(r)$, the pressure, $P(r)$, and the adiabatic exponent, $\gamma(r)$.** However, ρ and P are not independent, and related to each other through the hydrostatic equation:

$$\frac{dP}{dr} = -g\rho,$$

where

$$g = \frac{Gm}{r^2}, \;\; m = 4\pi \int_0^r \rho r'^2 dr'.$$

Therefore, only two thermodynamic (hydrostatic) properties of the Sun are independent, e.g. (ρ, γ), (P, γ), or their combinations: $(P/\rho, \gamma)$, (c^2, γ), (c^2, ρ) etc.

The general inverse problem in helioseismology is formulated in terms of small corrections to the standard solar model because the differences between the Sun and the standard model are typically 1% or less. When necessary the corrections can be applied repeatedly, using an iterative procedure.

Variational Principle: Rayleigh's Quotient

Consider the oscillation equations as a formal operator equation in terms of the vector displacement, $\vec{\xi}$:

$$\omega^2 \vec{\xi} = L(\vec{\xi}),$$

where L in the general case is an integro-differential operator. If we multiply this by $\vec{\xi}^*$ and integrate over the mass of the Sun we get:

$$\omega^2 \int_V \rho \vec{\xi}^* \cdot \vec{\xi} dV = \int_V \vec{\xi}^* \cdot L\vec{\xi} \rho dV,$$

where ρ_0 is the model density, V is the solar volume.
Then, the oscillation frequencies are:

$$\omega^2 = \frac{\int_V \vec{\xi}^* \cdot L\vec{\xi} \rho dV}{\int_V \rho \vec{\xi}^* \cdot \vec{\xi} dV}.$$

The frequencies are expressed in terms of **eigenfunctions** $\vec{\xi}$ and the solar properties represented by coefficients of operator L . Sometimes, this equation is called **Rayleigh's Quotient** (the original formulation: for an oscillatory system the averaged over period kinetic energy is equal the averaged potential energy).

Variational Principle

For small perturbations of the solar parameters the frequency change will depend on these perturbations and the corresponding perturbations of the eigenfunctions, e.g.

$$\delta\omega^2 = \Psi[\delta\rho, \delta\gamma, \delta\vec{\xi}\,].$$

The variational principle states that the perturbation of the eigenfunctions $\delta\vec{\xi}$ constitutes second-order corrections, that are in the first-order approximation:

$$\delta\omega^2 \approx \Psi[\delta\rho, \delta\gamma].$$

This allows us to neglect the perturbation of the eigenfunctions in the first-order perturbation theory.

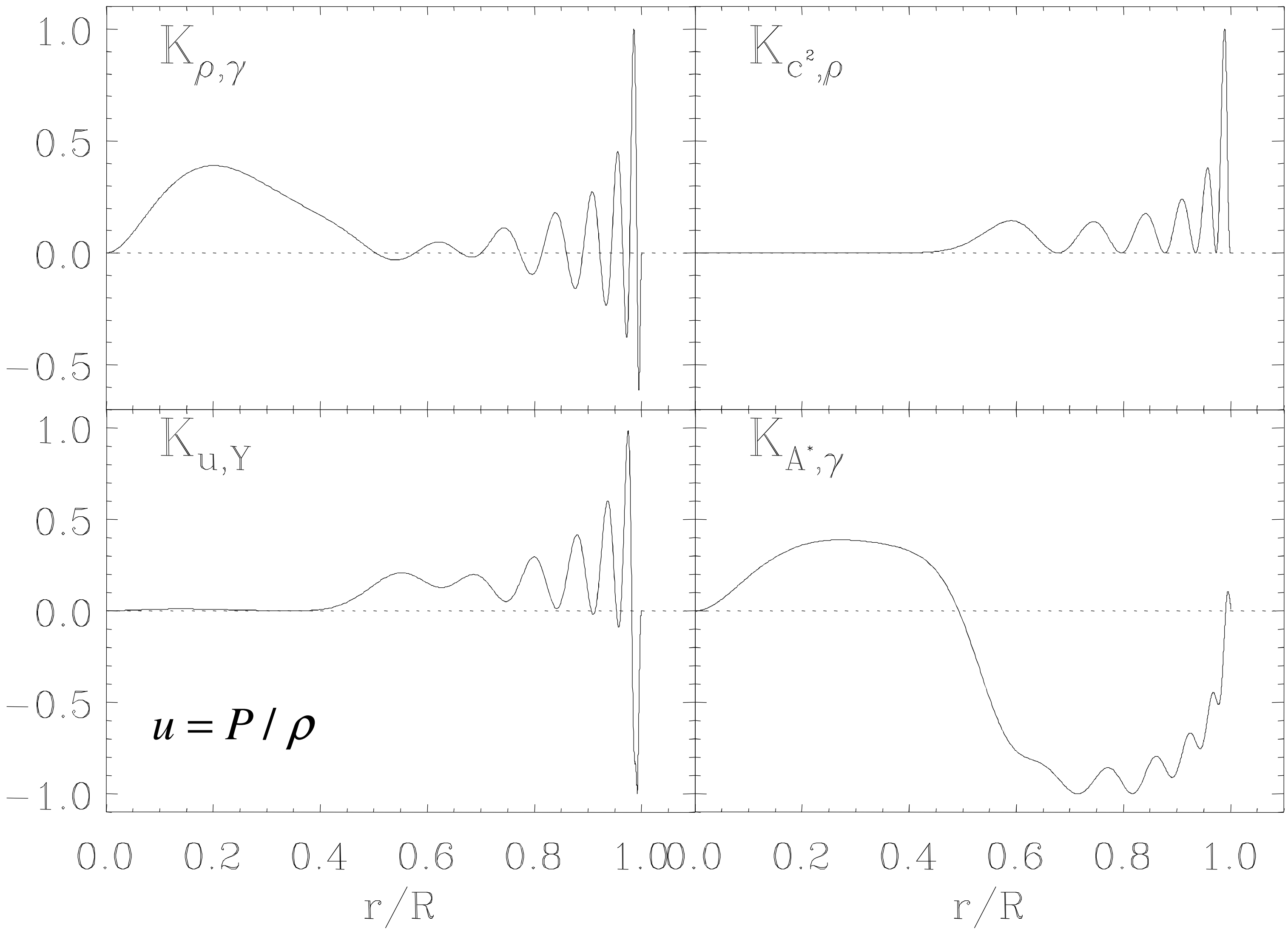
Sensitivity kernels

Using explicit formulations for operator L_1 the variational principle can be reduced to a system of integral equations for a chosen pair of independent variables, e.g. for (ρ,γ)

$$\frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}}=\int_0^R K_{\rho,\gamma}^{(n,l)}\frac{\delta\rho}{\rho}dr+\int_0^R K_{\gamma,\rho}^{(n,l)}\frac{\delta\gamma}{\gamma}dr,$$

where $K_{\rho,\gamma}^{(n,l)}(r)$ and $K_{\gamma,\rho}^{(n,l)}(r)$ are sensitivity (or 'seismic') kernels. These are calculated using the initial solar model parameters, ρ_0, P_0, γ, and the oscillation eigenfunctions for these model, $\vec{\xi}$.

A sample of the sensitivity kernels



Solution of the Inverse Problem

We have a system integral equations

$$\frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}} = \int_0^R K_{\rho,\gamma}^{(n,l)} \frac{\delta\rho}{\rho} dr + \int_0^R K_{\gamma,\rho}^{(n,l)} \frac{\delta\gamma}{\gamma} dr,$$

for a set of observed mode frequencies. If the number of observed frequencies is N (typically 2000), then we have a problem of determining two functions from this finite set. In general, it is impossible to determine these functions precisely. We can always find some rapidly oscillating functions, $f(r)$, that being added to the unknowns, $\delta\rho/\rho$ and $\delta\gamma/\gamma$, do not change the values of the integrals, e.g.

$$\int_0^R K_{\rho,\gamma}^{(n,l)}(r) f(r) dr = 0.$$

Such problems without an unique solution are called "ill-posed". The general approach is to find a smooth solution that satisfies the integral equations by applying some smoothness constraints to the unknown functions. This is called a "regularization procedure".

There are two basic methods for the helioseismic inverse problem:

1. Optimally Localized Averages (OLA) method - (Backus-Gilbert method)
2. Regularized Least-Squares (RLS) method - (Tikhonov method)

Optimally Localized Averages (OLA) method - (Backus-Gilbert method)

$$\frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}} = \int_0^R K_{\rho,\gamma}^{(n,l)} \frac{\delta\rho}{\rho} dr + \int_0^R K_{\gamma,\rho}^{(n,l)} \frac{\delta\gamma}{\gamma} dr,$$

$\frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}}$ - the relative difference between the observed and modeled frequencies is known from observations for a set of modes *(n,l)*. The frequencies are measured from the power spectrum.

Consider a linear combination of these equations with some unknown coefficients $a^{(n,l)}$ over the whole set of the observed modes:

$$\sum a^{(n,l)} \frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}} = \sum \int_0^R a^{(n,l)} K_{\rho,\gamma}^{(n,l)} \frac{\delta\rho}{\rho} dr + \sum \int_0^R a^{(n,l)} K_{\gamma,\rho}^{(n,l)} \frac{\delta\gamma}{\gamma} dr.$$

Change the order of summation and integration:

$$\sum a^{(n,l)} \frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}} = \int_0^R \sum a^{(n,l)} K_{\rho,\gamma}^{(n,l)} \frac{\delta\rho}{\rho} dr + \int_0^R \sum a^{(n,l)} K_{\gamma,\rho}^{(n,l)} \frac{\delta\gamma}{\gamma} dr.$$

Optimally Localized Averages Method

The idea of the OLA method is to find a linear combination of data such as the corresponding linear combination of the sensitivity kernels for one unknown will have an isolated peak at a given radial point, r_0, (resemble a δ-function), and the combination for the other unknown will be close to zero. Then this linear combination provides an estimate for the first unknown at r_0.

$$\sum a^{(n,l)} \frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}} = \int_0^R \sum a^{(n,l)} K_{\rho,\gamma}^{(n,l)} \frac{\delta\rho}{\rho} dr + \int_0^R \sum a^{(n,l)} K_{\gamma,\rho}^{(n,l)} \frac{\delta\gamma}{\gamma} dr.$$

If $\quad \sum a^{(n,l)} K_{\rho,\gamma}^{(n,l)}(r) \sim \delta(r - r_0), \quad$ and $\quad \sum a^{(n,l)} K_{\gamma,\rho}^{(n,l)}(r) \sim 0,$

then

$$\overline{\left(\frac{\delta\rho}{\rho}\right)}_{r_0} = \sum a^{(n,l)} \frac{\delta\omega^{(n,l)}}{\omega^{(n,l)}},$$

is an estimate of the density perturbation at $r = r_0$.

The coefficients, $a^{(n,l)}$, are different for different target radii r_0.

Averaging Kernels

The functions,

$$\sum a^{(n,l)} K_{\rho,\gamma}^{(n,l)}(r) \equiv A(r_0, r),$$

$$\sum a^{(n,l)} K_{\gamma,\rho}^{(n,l)}(r) \equiv B(r_0, r),$$

are called **"averaging kernels".**

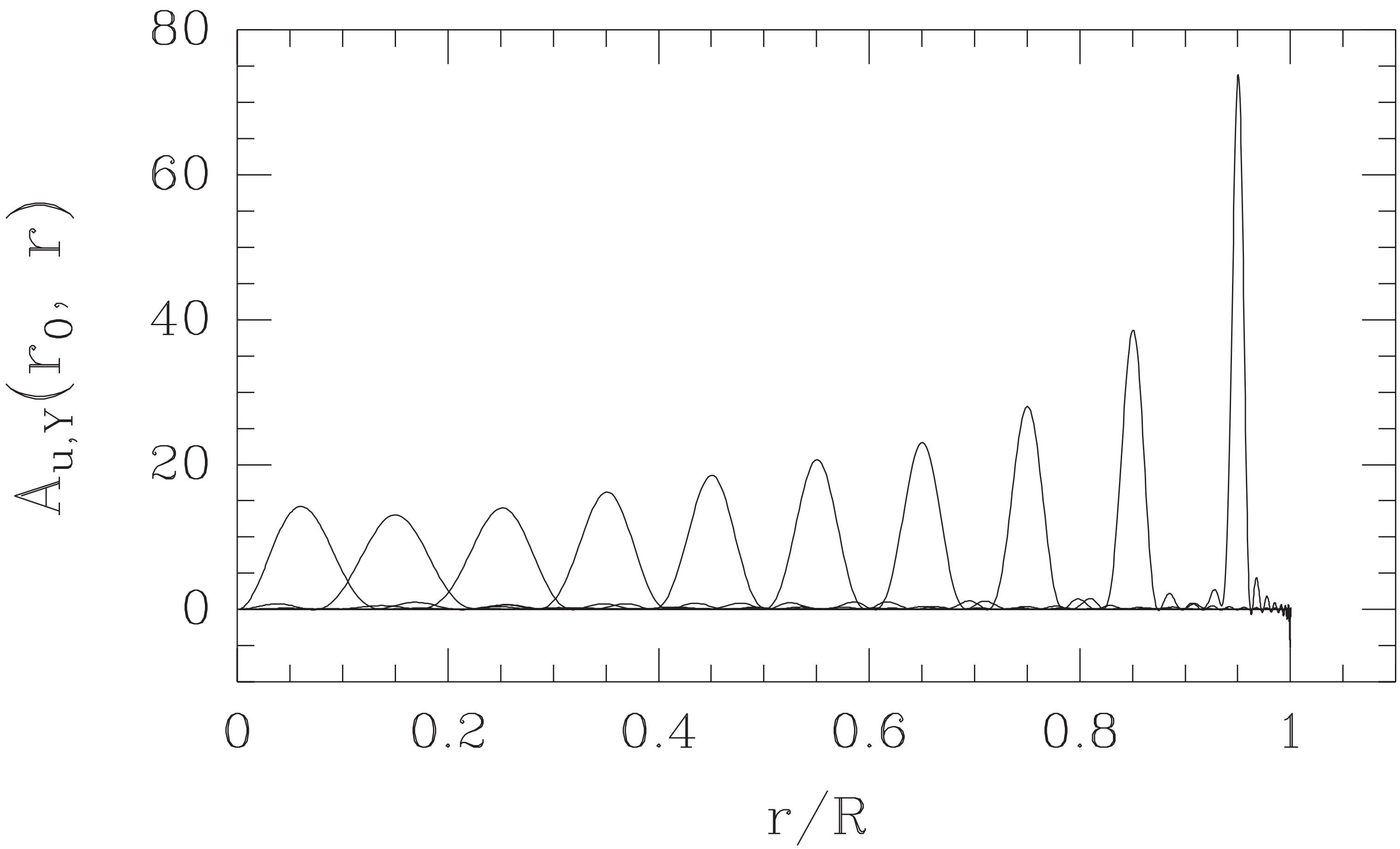
The coefficients, a^i , are determined my minimizing a quadratic form (here, we use index i instead of double index (n,l)):

$$M(r_0, A, \alpha, \beta) = \int_0^R J(r_0, r)\left[A(r_0, r)\right]^2 dr +$$

$$+\beta\int_0^R \left[B(r_0, r)\right]^2 dr + \alpha \sum_{i,j} E_{ij} a^i a^j,$$

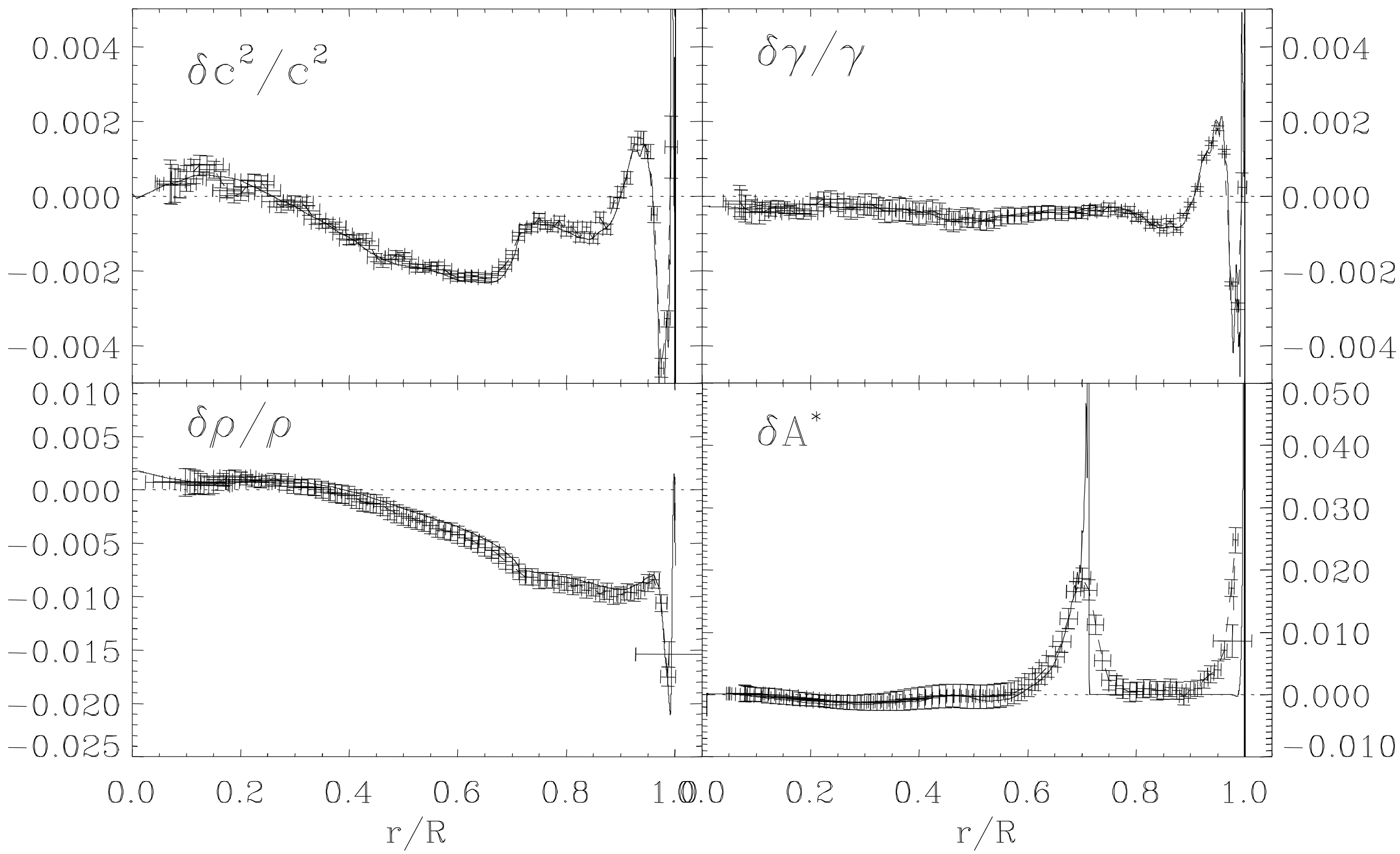
where $J(r_0, r) = 12(r - r_0)^2$, E_{ij} is a covariance matrix of observational errors, α and β are the regularization parameters. The first integral in this equation represents the Backus-Gilbert criterion of δ -ness for $A(r_0, r)$; the second term minimizes the contribution from $B(r_0, r)$, thus, effectively eliminating the second unknown function, ($\delta\gamma/\gamma$ in this case); and the last term minimizes the errors.

Example of optimally localized averaging kernels



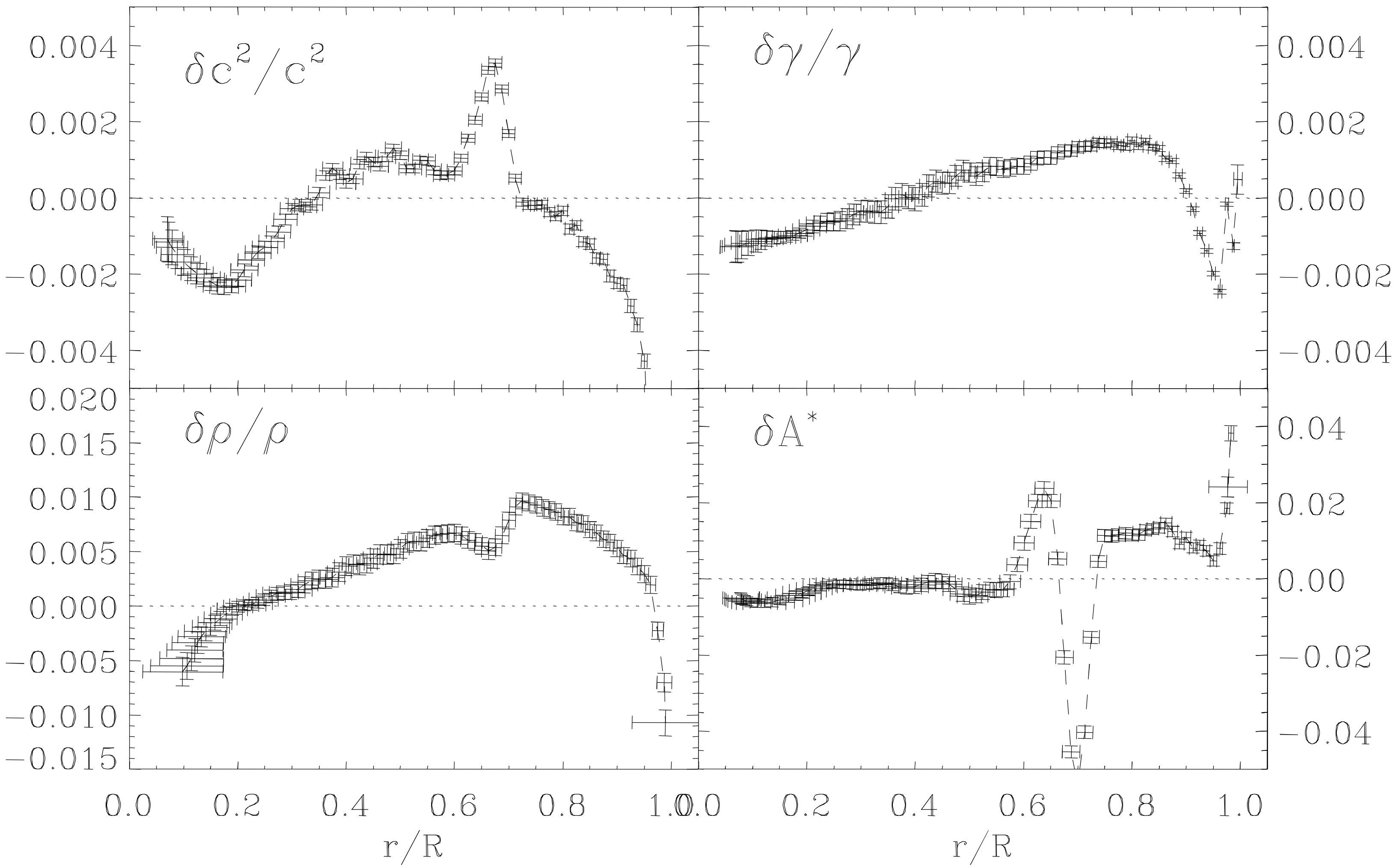
u=P/ρ, Y – helium abundance

Test inversions: between two different solar models

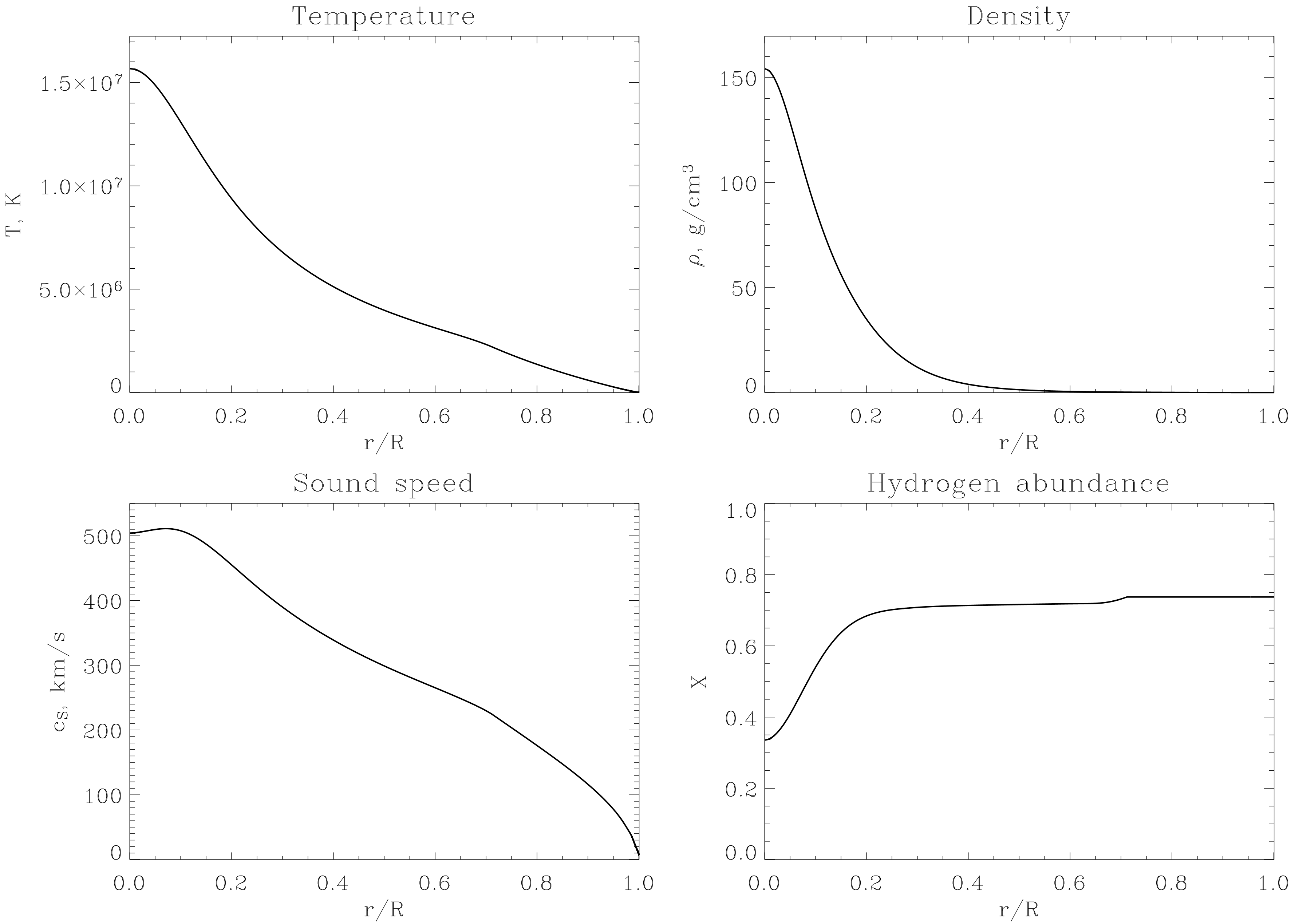


Crosses show the inversion results (horizontal bars – width of the averaging kernels).
Solid curves show the exact difference between the models.

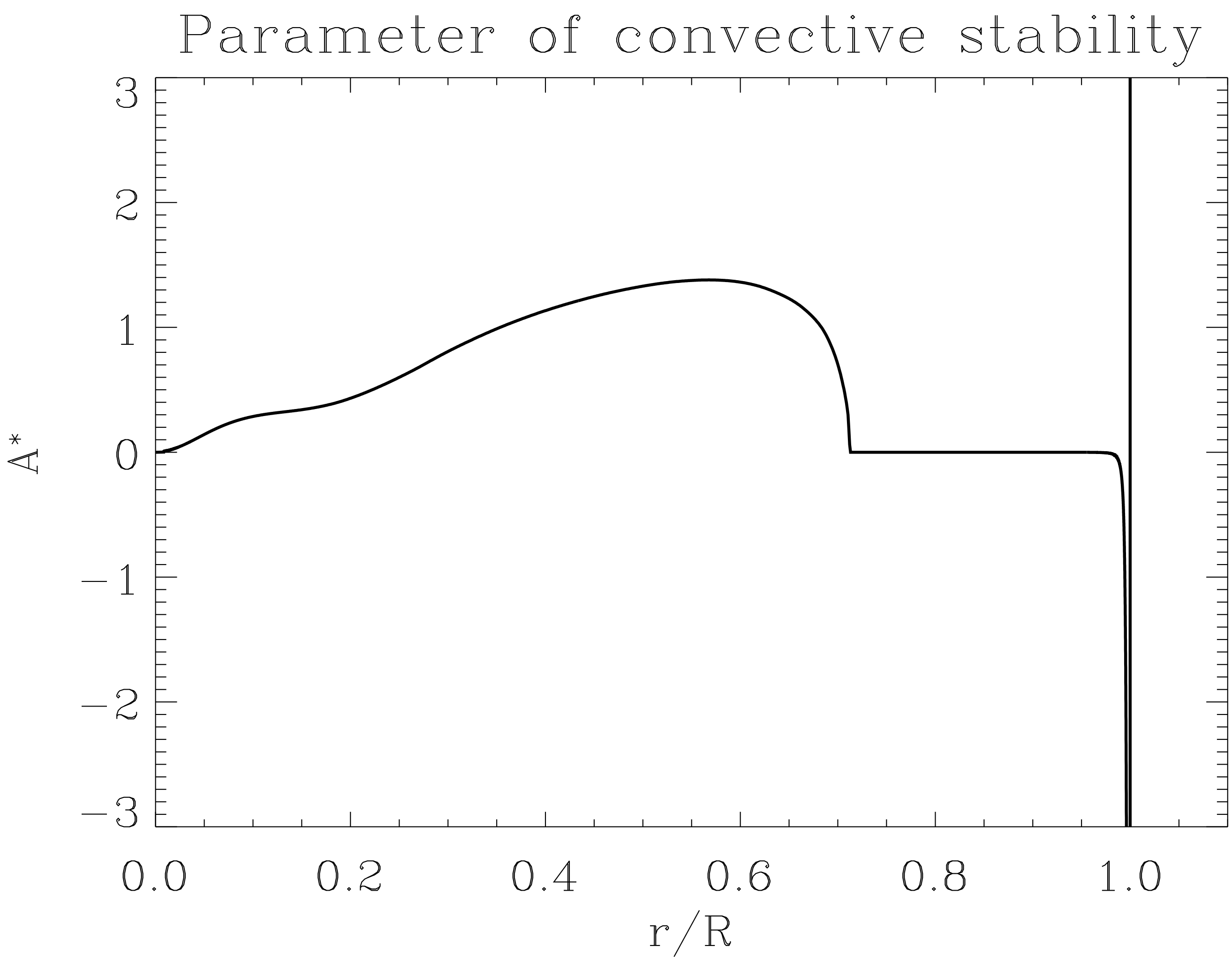
Inversion results for the observed solar frequencies



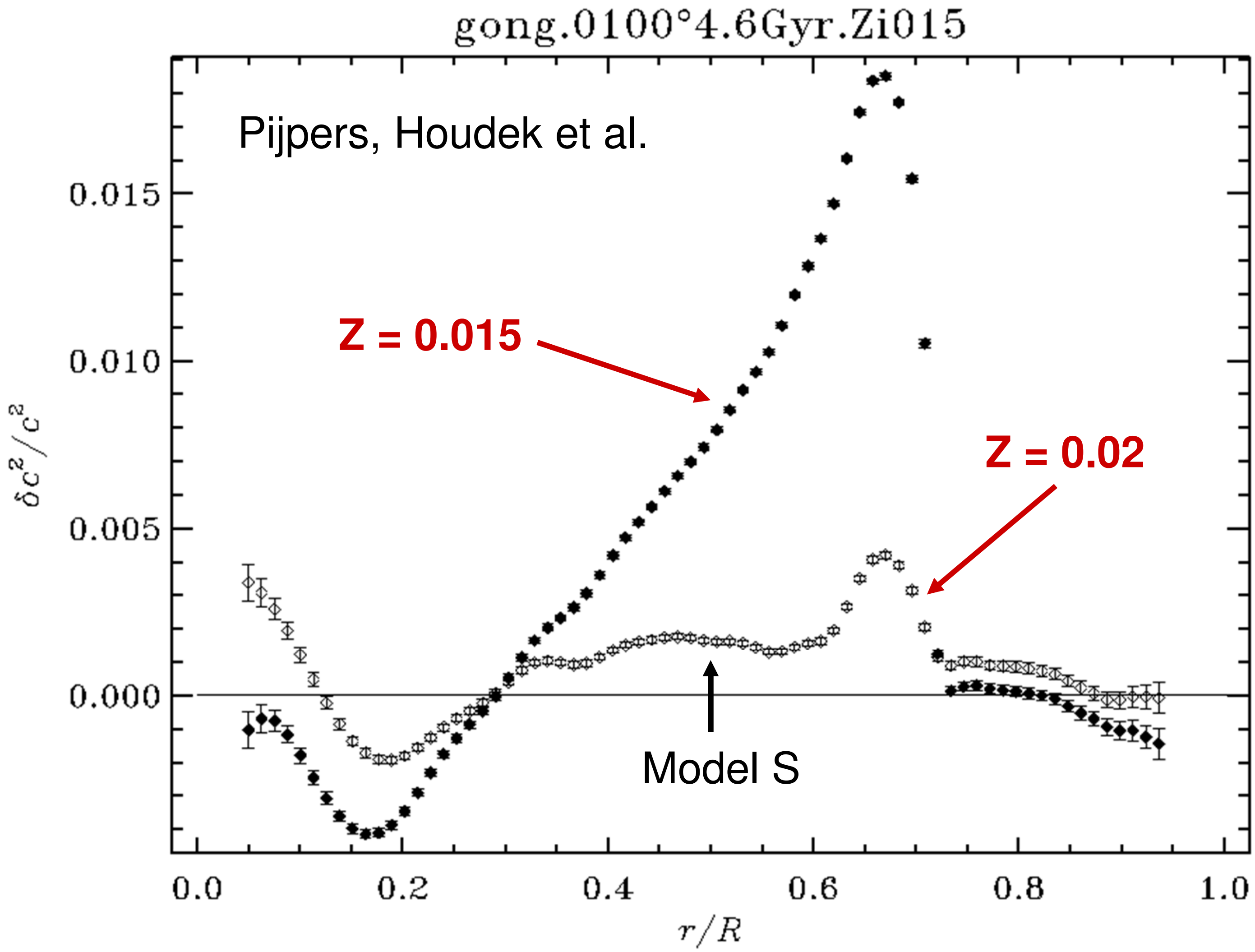
Standard solar model



Standard solar model



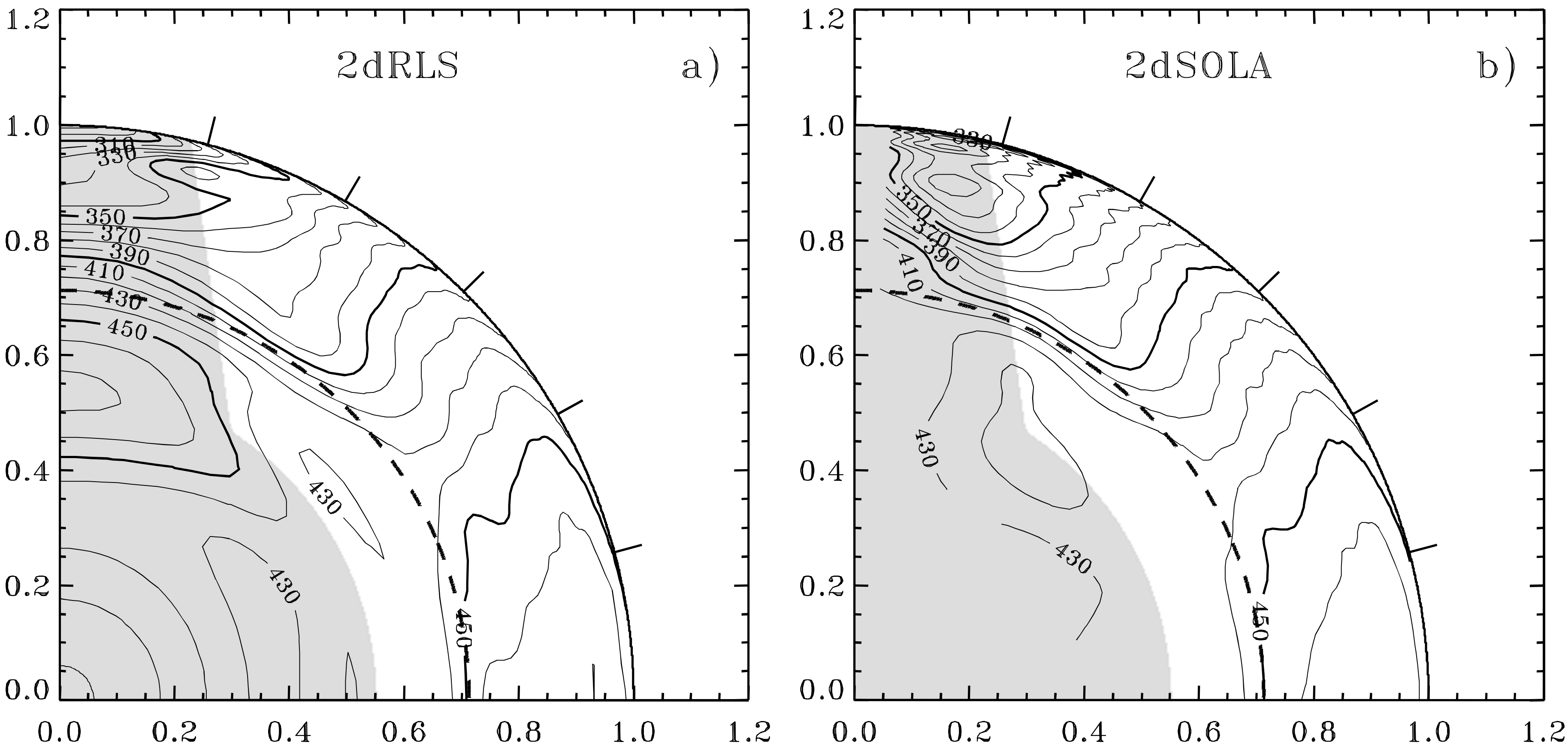
Revision of solar surface abundances

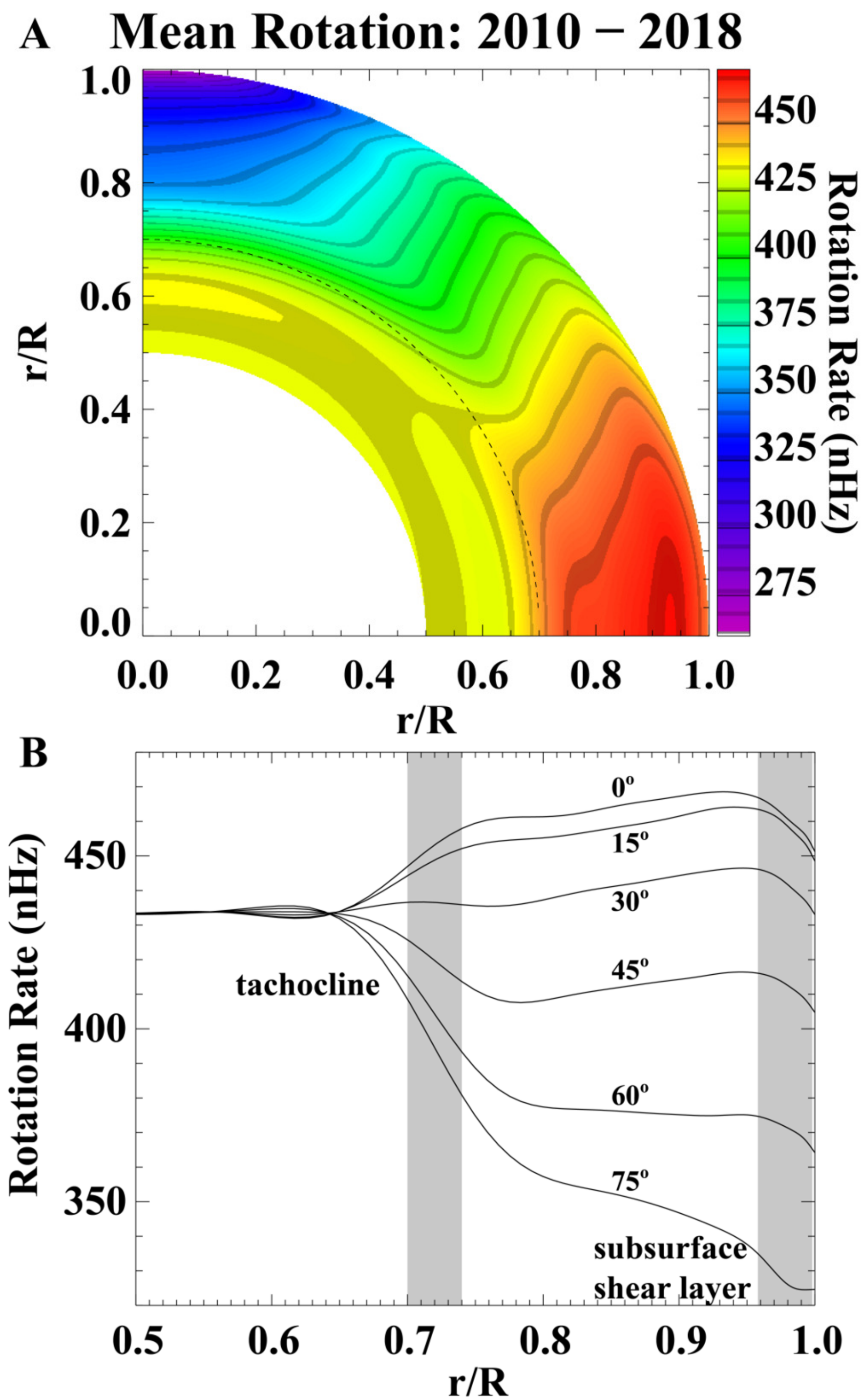


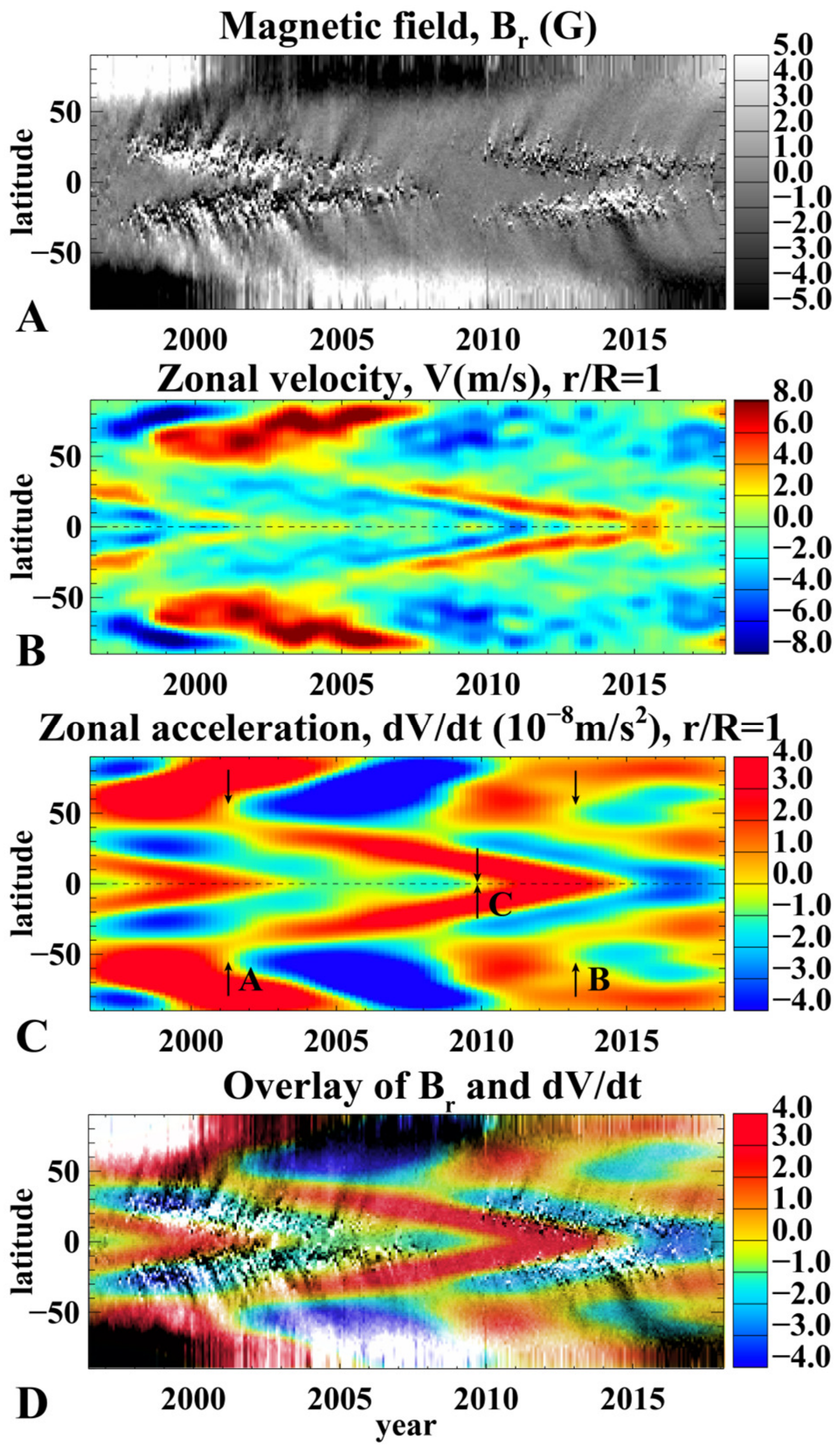
Helioseismic constraints on solar-cycle models

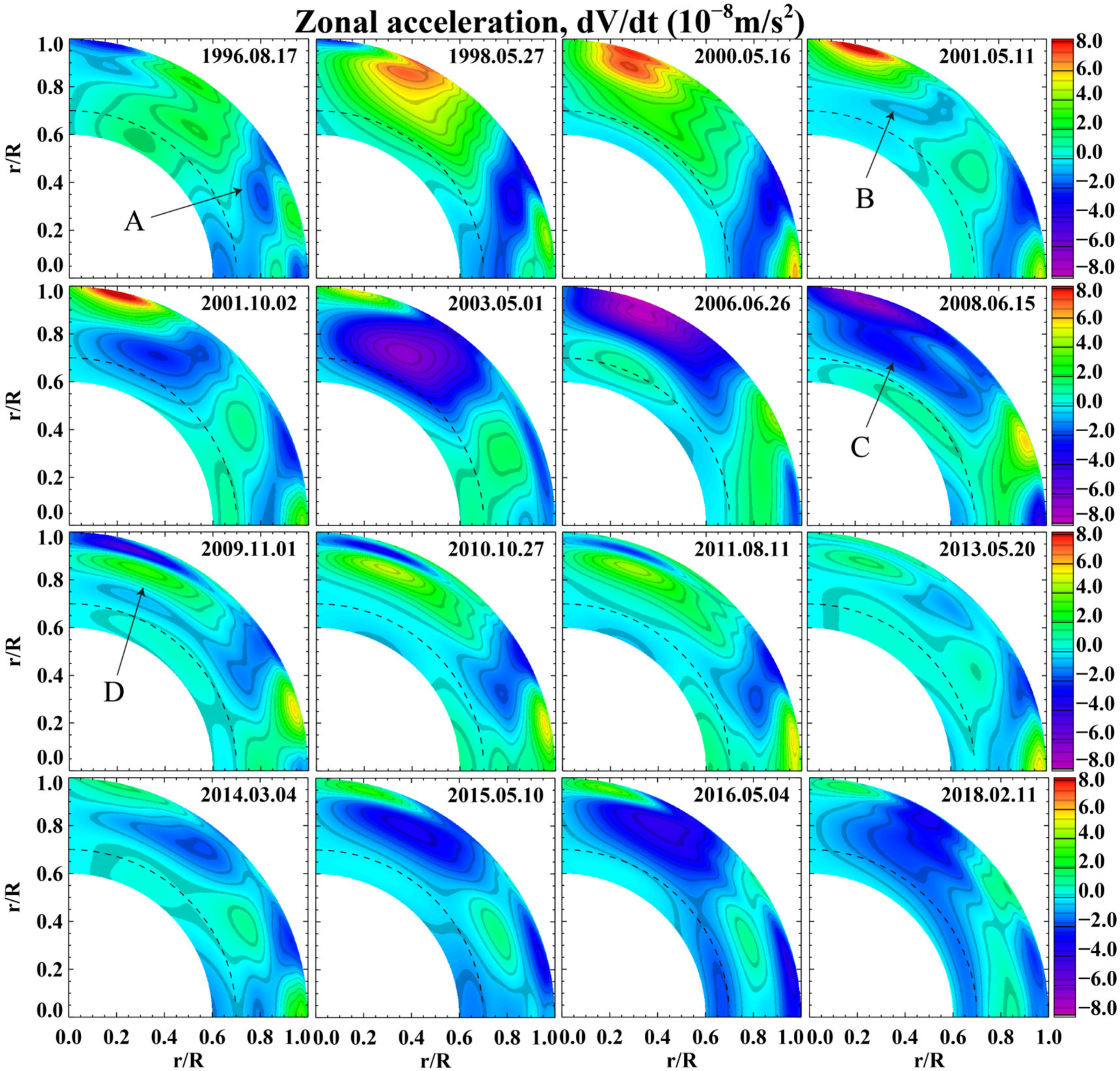
- Measurements of
  - Solar differential rotation
  - Torsional oscillations
  - Meridional circulation
  - Tachocline

Inversion of MDI data by two different techniques




A
Mean Rotation: 2010 − 2018
r/R
Rotation Rate (nHz)
450
425
400
375
350
325
300
275
1.0
0.8
0.6
0.4
0.2
0.0
B
Rotation Rate (nHz)
0°
15°
30°
45°
60°
75°
tachocline
subsurface
shear layer
0.5
0.6
0.7
0.8
0.9
1.0


Magnetic field, B_r (G)
5.0
4.0
3.0
2.0
1.0
0.0
−1.0
−2.0
−3.0
−4.0
−5.0
latitude
50
0
−50
A
2000
2005
2010
2015
Zonal velocity, V(m/s), r/R=1
8.0
6.0
4.0
2.0
0.0
−2.0
−4.0
−6.0
−8.0
B
Zonal acceleration, dV/dt (10^{-8}m/s^2), r/R=1
4.0
3.0
2.0
1.0
0.0
−1.0
−2.0
−3.0
−4.0
A
B
C
C
Overlay of B_r and dV/dt
D
year


Zonal acceleration, dV/dt (10^{-8}m/s^2)
1996.08.17
1998.05.27
2000.05.16
2001.05.11
2001.10.02
2003.05.01
2006.06.26
2008.06.15
2009.11.01
2010.10.27
2011.08.11
2013.05.20
2014.03.04
2015.05.10
2016.05.04
2018.02.11
A
B
C
D
r/R
0.0 0.2 0.4 0.6 0.8 1.0
8.0 6.0 4.0 2.0 0.0 −2.0 −4.0 −6.0 −8.0