15. Solar MHD II.

Solar MHD

- Particle Motion in Electric Field
- Magnetic Effects
- Ohm’s Law
- MHD Equations
- Magnetic Field Diffusion
- Frozen Magnetic Flux Approximation
- Magnetic Forces
- MHD Waves
- *Magnetohydrodynamics movie:* https://www.youtube.com/watch?v=QArcTylNooQ

MHD Equations

Consider plasma in an electro-magnetic field.
The Maxwell equations are (CGS units):

MHD approximation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \cancel{\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}},$$

neglect displacement current because MHD processes are slow compared to the speed of light

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = \cancel{4\pi\rho_e},$$

neglect separation of electric charges of electrons and ions – plasma quasi-neutrality

$\mathbf{j} = \sigma(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$ - Ohm’s law in a non-magnetized plasma $(\omega_L \tau \ll 1)$ moving with velocity **v;** recall the electric field transformation in a moving coordinate system: $\boldsymbol{E}' = \boldsymbol{E} + \frac{1}{c}\left(\mathbf{v} \times \boldsymbol{B}\right)$

Equations of conservation of mass, momentum and energy

Then, we combine the Maxwell equations with the equations of conservation of mass, momentum and energy:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0,$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right) = -\nabla P + \frac{1}{c}\mathbf{j}\times\mathbf{B},$$

$$\rho T \frac{dS}{dt} = Q - L,$$

where ρ is the mass density, P is the gas pressure, $\mathbf{v}$ is the velocity, $S = c_V \log(P/\rho^{\gamma})$ is the specific entropy, Q is an energy input, L is the energy loss rate (radiative losses).

Magnetic Field Diffusion

Applying curl to the Ohm's equation:

$$\mathbf{E} = \frac{1}{\sigma}\mathbf{j} - \frac{1}{c}(\mathbf{v}\times\mathbf{B}),$$

$$\nabla\times\mathbf{E} = \frac{1}{\sigma}\nabla\times\mathbf{j} - \frac{1}{c}\nabla\times(\mathbf{v}\times\mathbf{B}),$$

and using the Maxwell equations:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$

we get an equation for magnetic field strength $\mathbf{B}$:

$$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\boldsymbol{E} = -c\left(\frac{1}{\sigma}\nabla\times\mathbf{j} - \frac{1}{c}\nabla\times(\mathbf{v}\times\mathbf{B})\right),\ \text{or}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \frac{c^2}{4\pi}\nabla\times\left(\frac{1}{\sigma}\nabla\times\mathbf{B}\right).$$

This is "the induction equation" - a central equation for solar MHD theories.

Magnetic Reynolds Number

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{c^2}{4\pi} \nabla \times \left(\frac{1}{\sigma} \nabla \times \mathbf{B} \right).$$

advection of magnetic field

magnetic field diffusion

The first term in the right-hand side describes advection of magnetic field, the second term corresponds to magnetic field diffusion due to Joule dissipation.

The relative importance of these terms for a process of a characteristic scale L, velocity v is determined by the magnetic Reynolds number $\mathbf{R_M}$ or $\mathbf{Re_M}$:

$$R_M = \frac{\frac{vB}{L}}{\frac{c^2}{4\pi\sigma}\frac{B}{L^2}} = \frac{4\pi\sigma L v}{c^2}.$$

For typical coronal conditions: $T = 10^6\,\mathrm{K}$, $\sigma = 10^{12}\ \mathrm{s}^{-1}$, $L = 10^8\ \mathrm{cm}$, $v = 10^5\ \mathrm{cm/s}$,

$$R_M \sim 10^4 >> 1.$$

Coefficient of magnetic diffusion

For uniform σ the last term can be simplified:

$$\nabla\times(\nabla\times\mathbf{B})=\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=-\nabla^2\mathbf{B}.$$

Then, if $\mathbf{v}=0$ we get a diffusion equation: $$\frac{\partial\mathbf{B}}{\partial t}=D\nabla^2\mathbf{B},$$

where $D=\dfrac{c^2}{4\pi\sigma}$ is a diffusion coefficient for magnetic field.

Exercises:

1. Estimate the characteristic scale of dissipation of magnetic field in solar flares. The duration of solar flares is 10^3 sec.

$$L\sim\sqrt{\frac{c^2t}{4\pi\sigma}}\sim10^5\,\text{cm}=1\text{km}.$$

2. This is smaller than the observed flare structure. What does that mean?
3. Estimate the decay time of sunspots ($L\sim10^9$ cm, $T\sim10^4$ K, $\sigma\sim10^9\ \text{s}^{-1}$).

$$t\sim\frac{4\pi\sigma L^2}{c^2}\sim10^7\ \text{sec}\sim4\ \text{months}.$$

4. This is longer than the observed lifetime of sunspots. Why?

Ideal MHD approximation

The MHD equations without energy input and dissipation are called the `ideal MHD' approximation:

$$\frac{d\rho}{dt} + \rho \nabla \mathbf{v} = 0$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \frac{1}{4\pi} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma} \right) = 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{B})$$

The ideal MHD approximation is often used for modeling processes with high magnetic Reynolds number.

Linearized ideal MHD equations are used to describe MHD waves.
Consider small perturbations of velocity, density, pressure and magnetic field:
$\boldsymbol{v},\ \rho',\ P',\ \boldsymbol{b}$ in uniform stationary plasma with constant magnetic field:
velocity $\boldsymbol{v}_0 = 0,$ density $\rho_0,$ magnetic field $\boldsymbol{B}_0.$

Frozen Magnetic Flux Approximation

Consider a high-conductivity plasma, $R_M \gg 1$, or $\sigma = \infty$ ('ideal plasma'). Then the equations for magnetic field are: $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$

$$\nabla \cdot \mathbf{B} = 0.$$

Consider a 1D case: $\mathbf{B} = (0, B, 0)$, $\mathbf{v} = (v, 0, 0)$, - plasma motion across the magnetic field lines, with B and v depending only on x: $\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}(vB).$

From this and the mass equation: $\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v),$

we get

$$\frac{dB}{dt} + B\frac{\partial v}{\partial x} = 0,$$

$$\frac{d\rho}{dt} + \rho\frac{\partial v}{\partial x} = 0.$$

Then

$$\frac{dB}{dt} - \frac{B}{\rho}\frac{d\rho}{dt} = 0,$$

or $\frac{d\left(\frac{B}{\rho}\right)}{dt} = 0,$ or $\frac{B}{\rho} = \text{const}.$

For a general 3D case: $\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\nabla\right)\mathbf{v}.$

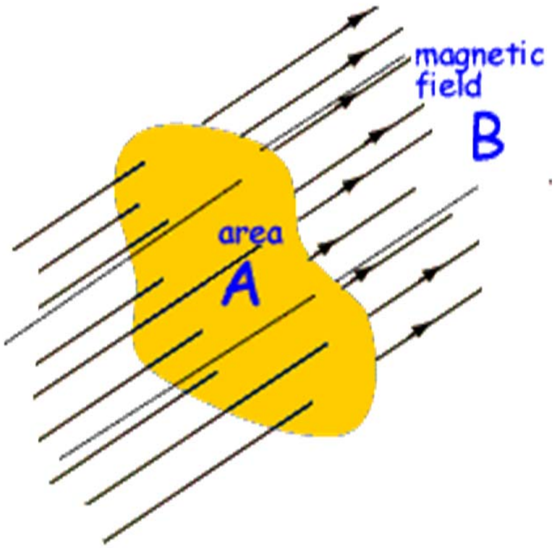
This equation shows that in ideal plasma magnetic field is coupled with plasma density. It follows the plasma motions.

Magnetic Flux Conservation

Consider magnetic flux Φ through a plasma area S restricted by a closed curve Γ:

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{s},$$

where $\mathbf{s}$ is a vector perpendicular to the area.



If we change the contour line of this plasma element then the total flux will change due to:

1) the change of the magnetic field strength (as follows from the MHD equations);

2) the change of the area of this element:

$$\frac{d\Phi}{dt} = \frac{d\Phi'}{dt} + \frac{d\Phi''}{dt},$$

where
$$\frac{d\Phi'}{dt} = \iint_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{s}.$$

Now consider a small change of the area $d\mathbf{s}$ due to plasma motion with velocity $\mathbf{v}$ during time dt:

$$d\mathbf{s} = \mathbf{v}dt \times d\mathbf{l},$$

where $\mathbf{l}$ is the change of the length of contour Γ.
Then, the change of magnetic flux IS:

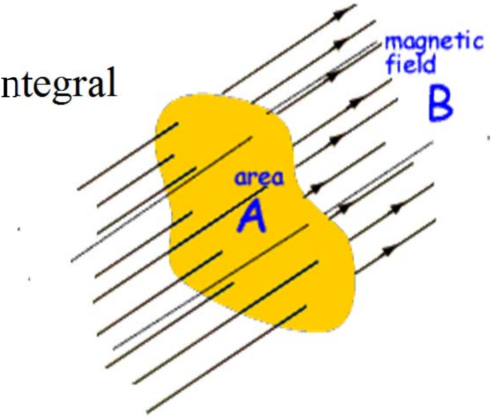
$$d\Phi'' = \mathbf{B} \cdot d\mathbf{s} = \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l}dt = -dt(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

where we used a vector-product relation.

Then, $$\frac{d\Phi''}{dt} = -\oint_\Gamma \mathbf{v} \times \mathbf{B} d\mathbf{l}.$$ $$\frac{d\Phi}{dt} = \iint_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{s} - \oint_\Gamma \mathbf{v} \times \mathbf{B} d\mathbf{l}.$$

Magnetic Flux Conservation

$$\frac{d\Phi}{dt} = \iint_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{s} - \oint_\Gamma \mathbf{v}\times\mathbf{B}d\mathbf{l},$$



Using Stokes' theorem to replace the contour integral with the surface integral

$$\oint_\Gamma \mathbf{v}\times\mathbf{B}d\mathbf{l} = \iint_S \left[\nabla\times(\mathbf{v}\times\mathbf{B})\right]\cdot d\mathbf{s},$$

we obtain for the total flux:

$$\frac{d\Phi}{dt} = \iint_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B})\right] d\mathbf{s} = 0.$$

The right-hand side is equal zero because it satisfies the equation for magnetic field in an ideal plasma.

Thus, $\frac{d\Phi}{dt} = 0$.

This is **<u>the frozen flux theorem</u>: the total magnetic flux through a plasma element does not change under deformations of this element.**

This can be interpreted as magnetic field lines move with the plasma ("frozen into the plasma").

Magnetic Forces

The Lorentz force, $\frac{1}{c}\mathbf{j}\times\mathbf{B}$**,** in the momentum equation can be expressed in terms of magnetic field using the Maxwell equation, $\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}$:

$$\mathbf{f}=\frac{1}{4\pi}\nabla\mathbf{B}\times\mathbf{B}.$$

Using the standard vector formula: $\frac{1}{2}\nabla\mathbf{a}^2=(\mathbf{a}\nabla)\mathbf{a}+\mathbf{a}\times\nabla\mathbf{a},$

$$\mathbf{f}=-\frac{1}{8\pi}\nabla B^2+\frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{B}=-\nabla P_M+\frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{B}.$$

The first term is a gradient of magnetic pressure: $P_M=B^2/8\pi$;
the second term describes magnetic tension force.
The tension force is analogous to restoring force of rubber bands.

The magnetic forces can be written in a tensor form:

$$\mathbf{f}=-\frac{\partial}{\partial x_k}T_{ik},$$

where $T_{ik}=\frac{1}{4\pi}\left(\frac{1}{2}\delta_{ik}B^2-B_iB_k\right)$ are Maxwell stresses. The magnetic forces are anisotropic.

MHD Waves

Magnetic forces can produce additional restoring force to small perturbations in a magnetized plasma and cause oscillations.

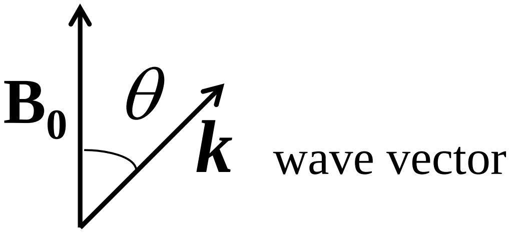
Consider a linearized system of the ideal MHD equations for perturbations ρ', velocity $\mathbf{v}$, and magnetic field, $\mathbf{b}$:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \mathbf{v} = 0.$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla P' + \frac{1}{4\pi} \nabla \times \mathbf{b} \times \mathbf{B}_0$$

$$P' = \frac{\gamma P_0}{\rho_0} \rho' = c_S^2 \rho' \quad (c_S \text{ is the adiabatic sound speed})$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0)$$



Consider a periodic solution: $\rho', \mathbf{v}, \mathbf{b} \propto e^{i\mathbf{kr} - i\omega t}$

for $\mathbf{k} = (0, 0, k)$, $\mathbf{B}_0 = (0, B_0 \sin\theta, B_0 \cos\theta)$ - wave propagation along z-axis;

θ (the angle between $\mathbf{k}$ and $\mathbf{B}_0$).

If $B_0 = 0$ we obtain the dispersion relation for ordinary sound waves: $\omega^2 = c_S^2 k^2$.

The phase speed: $u = \omega / k = c_{S.}$

Alfven waves

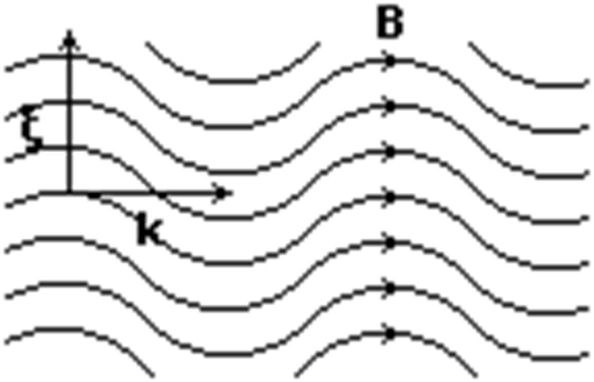
If $B_0 \neq 0$ then we have solutions (dispersion relations) of two types:

1) $$\omega^2 = k^2 \frac{B^2}{4\pi\rho} \cos^2\theta,$$

- Alfven waves; $v_A = B/\sqrt{4\pi\rho}$ **is the Alfven speed. These waves are incompressible** $\rho' = 0$**.**

Plasma moves perpendicular to magnetic field lines.

The phase speed $u = \omega/k$ is: $u_{Alfven} = \pm v_A \cos\theta.$



Alfvén

Fast and slow MHD waves

$$2)\quad u^2 = \omega^2 / k^2 = \frac{1}{2}(c_S^2 + v_A^2) \pm \frac{1}{2}\sqrt{v_A^4 + c_S^4 - 2v_A^2 c_S^2 \cos 2\theta},$$

where c_S is the sound speed.

The solution with "+" is called fast MHD wave, with "-" - slow MHD wave.

For $\theta = 0$ (propagation along field lines) and:

- $v_A > c_S:\ u_{fast} = v_A,\quad u_{slow} = c_S$ - strong magnetic field
- $v_A < c_S:\ u_{fast} = c_S,\quad u_{slow} = v_A$ - weak magnetic field

for $\theta = \pi/2$ (propagation perpendicular to the field lines):

- $u_{fast} = \pm\sqrt{v_A^2 + c_S^2},\quad u_{slow} = 0$

In the fast MHD waves the magnetic and pressure forces act together, in the slow MHD waves they act against each other.

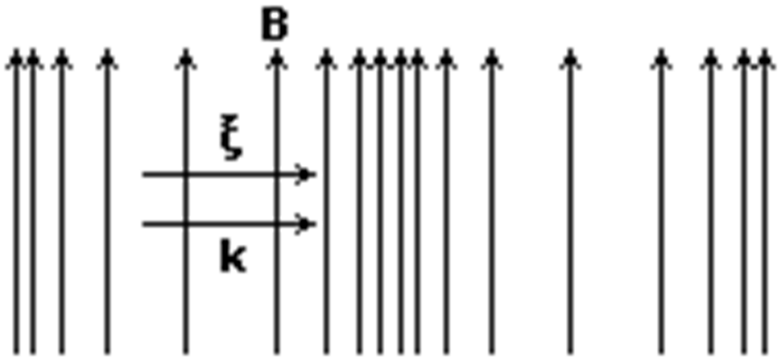
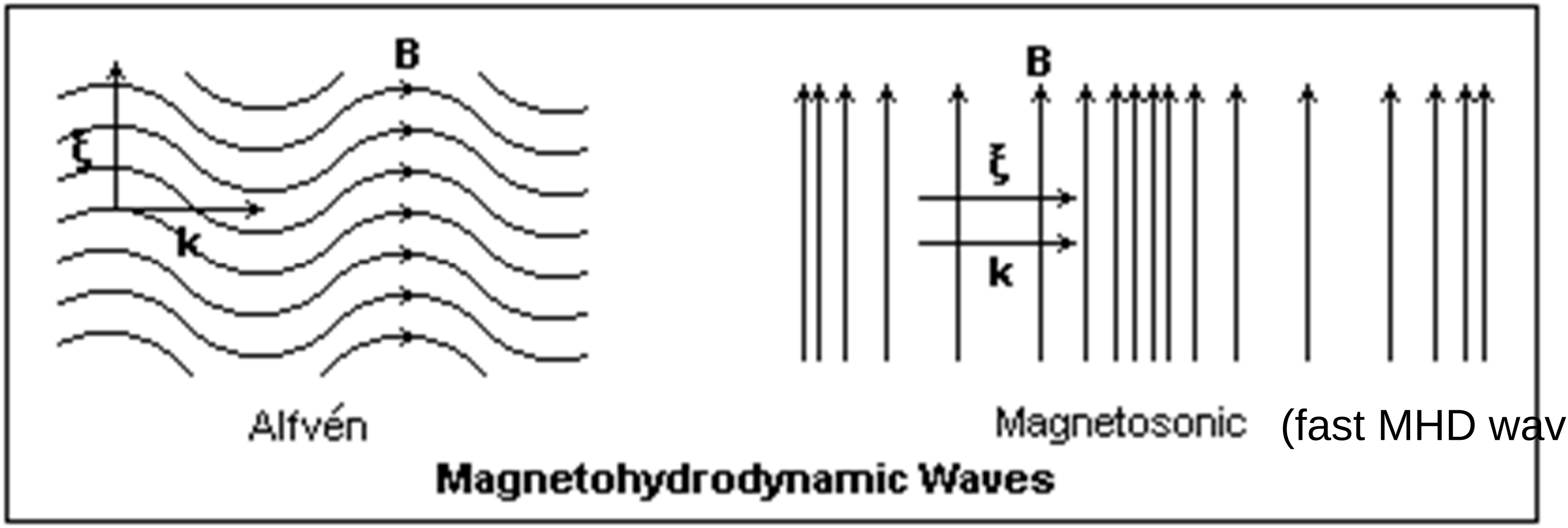


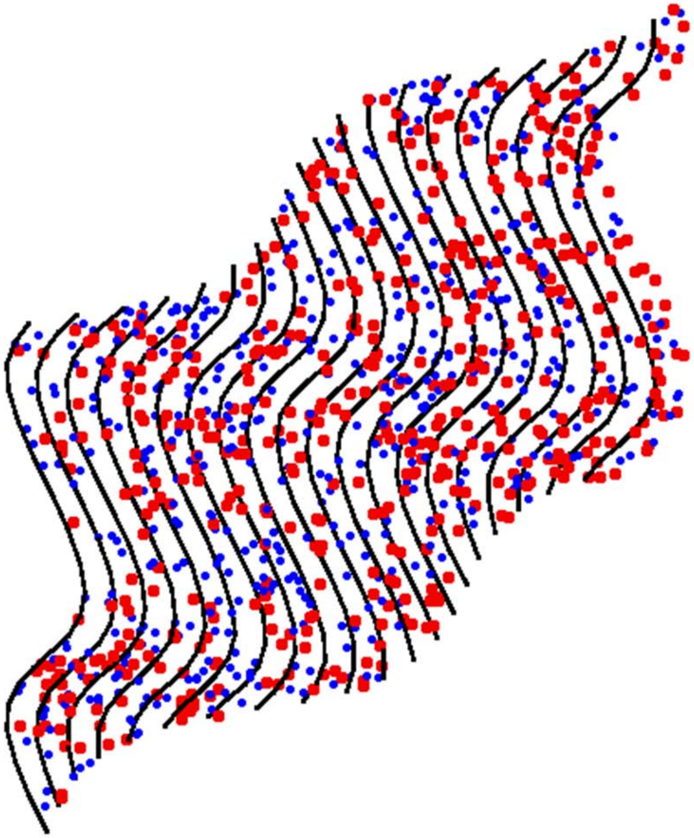
Illustration of Alfven and magnetosonic (fast MHD) waves



(fast MHD waves)

- **The Alfven waves** are incompressible. They transfer vorticity along the field lines. Plasma oscillates across the initial field lines.
- **The fast MHD waves** mostly travel across the magnetic field lines with a speed higher the speed of sound and the Alfven speed.
- **The slow MHD waves** mostly travel along the field line with a speed slower than the sound speed.
- In plasma of variable density (solar atmosphere) the waves can transform from one type to another.

Animation of Alfven wave – magnetic field lines are frozen in plasma



Hinode observations of Alfven waves in the solar corona

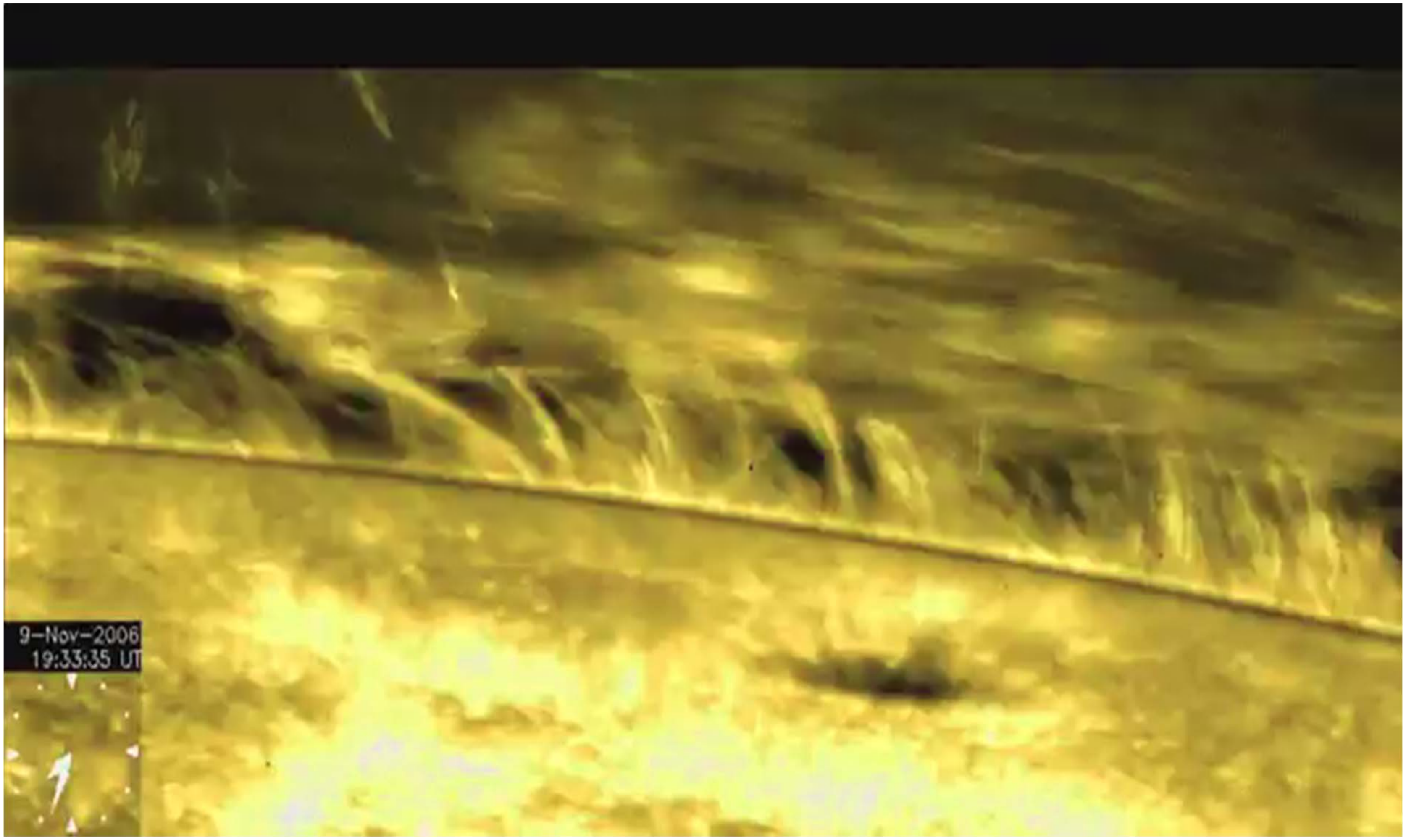
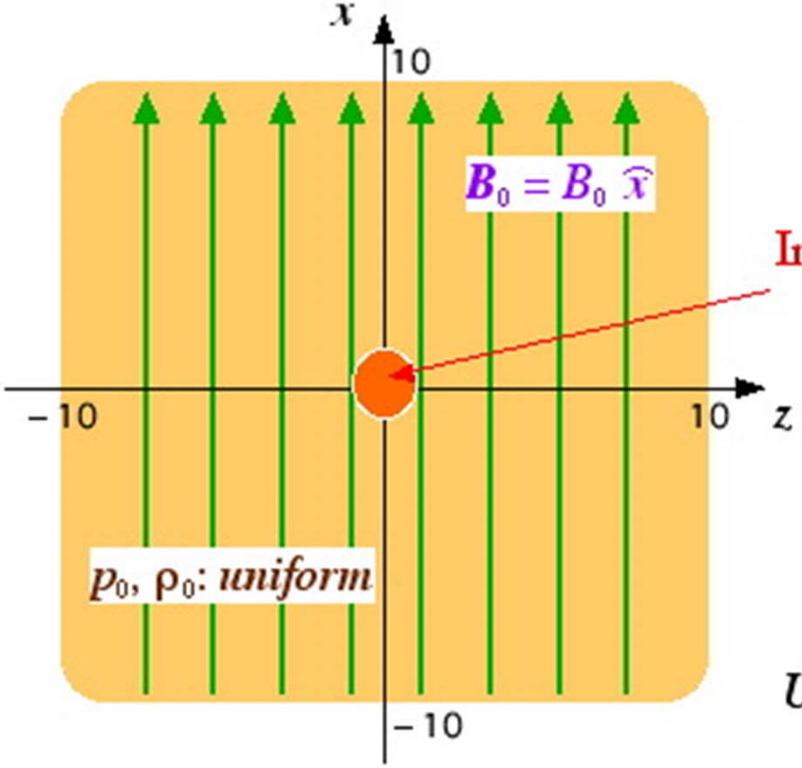


Illustration of the wave propagation and energy transport from a localized pressure perturbation (T. Magara)

Procedure of numerical simulation:

2-dimensional, adiabatic MHD simulation



Input initial perturbation:

$$p_0 \rightarrow p_0 + 0.1\,p_0$$

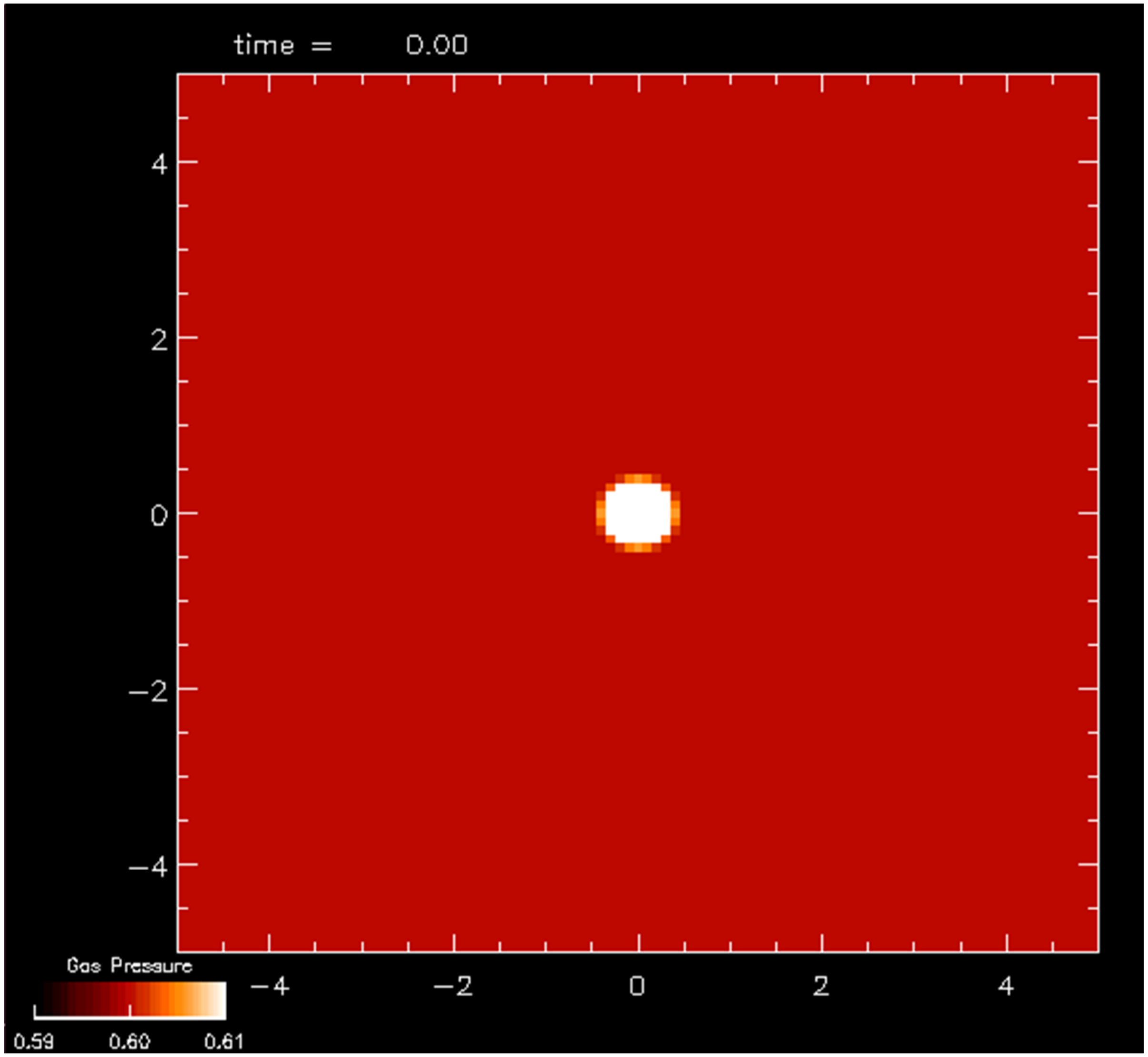
$$p(x, z) = p_0\left[0.05\left\{\cos(2\pi r) + 1\right\} + 1\right]$$

for $r \equiv \sqrt{x^2 + z^2} < 0.5$

Unit of velocity:

$$c_s = \sqrt{\gamma\,\frac{p_0}{\rho_0}} = 1$$

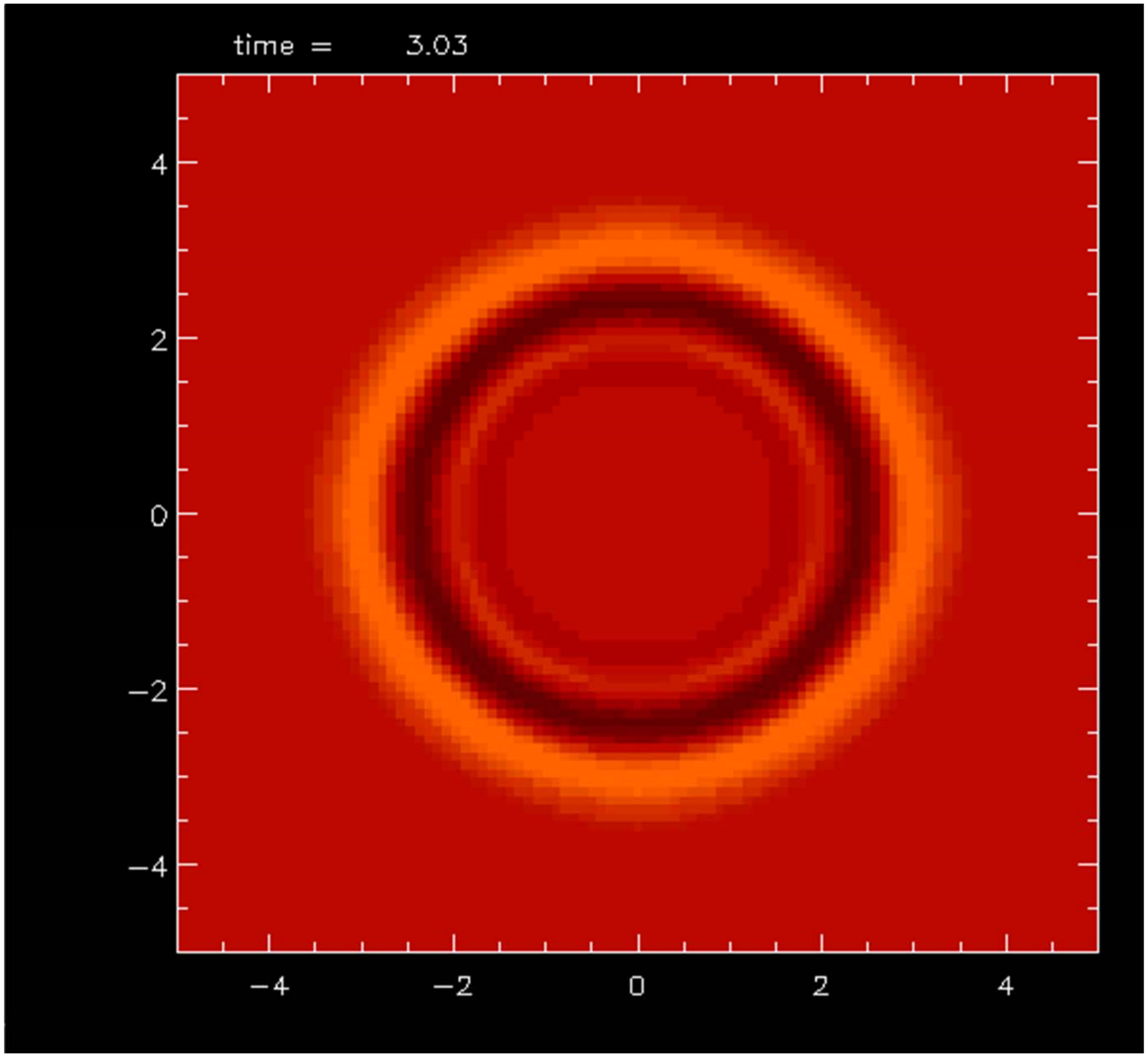
Sound wave (B_0=0)



Pressure perturbation

Without magnetic field the solution represents an isotropic sound wave.

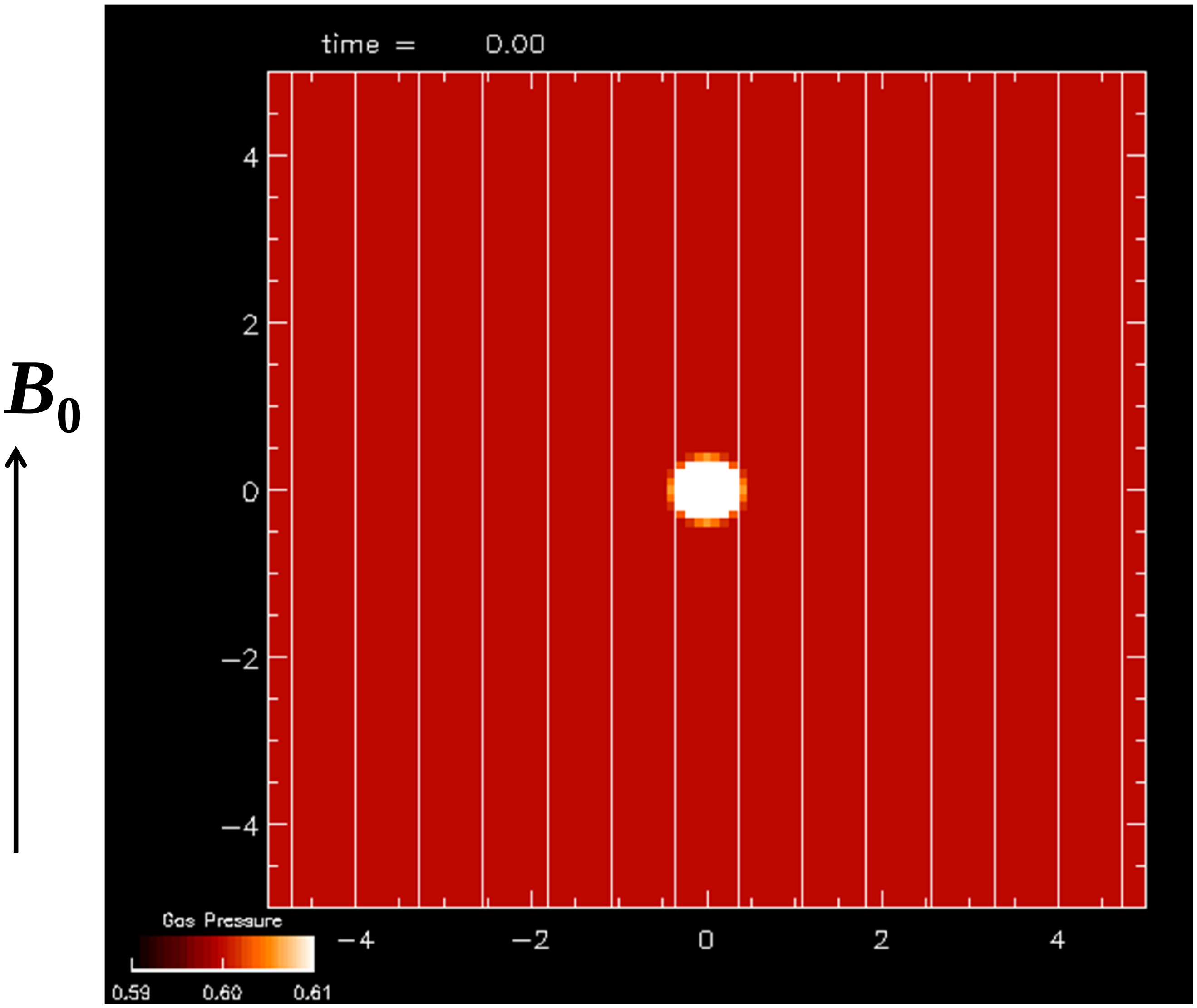
Sound wave (B_0=0)



Pressure perturbation

Without magnetic field the solution represents an isotropic sound wave.

Weak magnetic field ($c_S > v_A$)



Pressure perturbation

In the case of weak field:

$$c_S > c_A, \text{ or}$$

$$\frac{\gamma P}{\rho} > \frac{B^2}{4\pi\rho}, \text{ or}$$

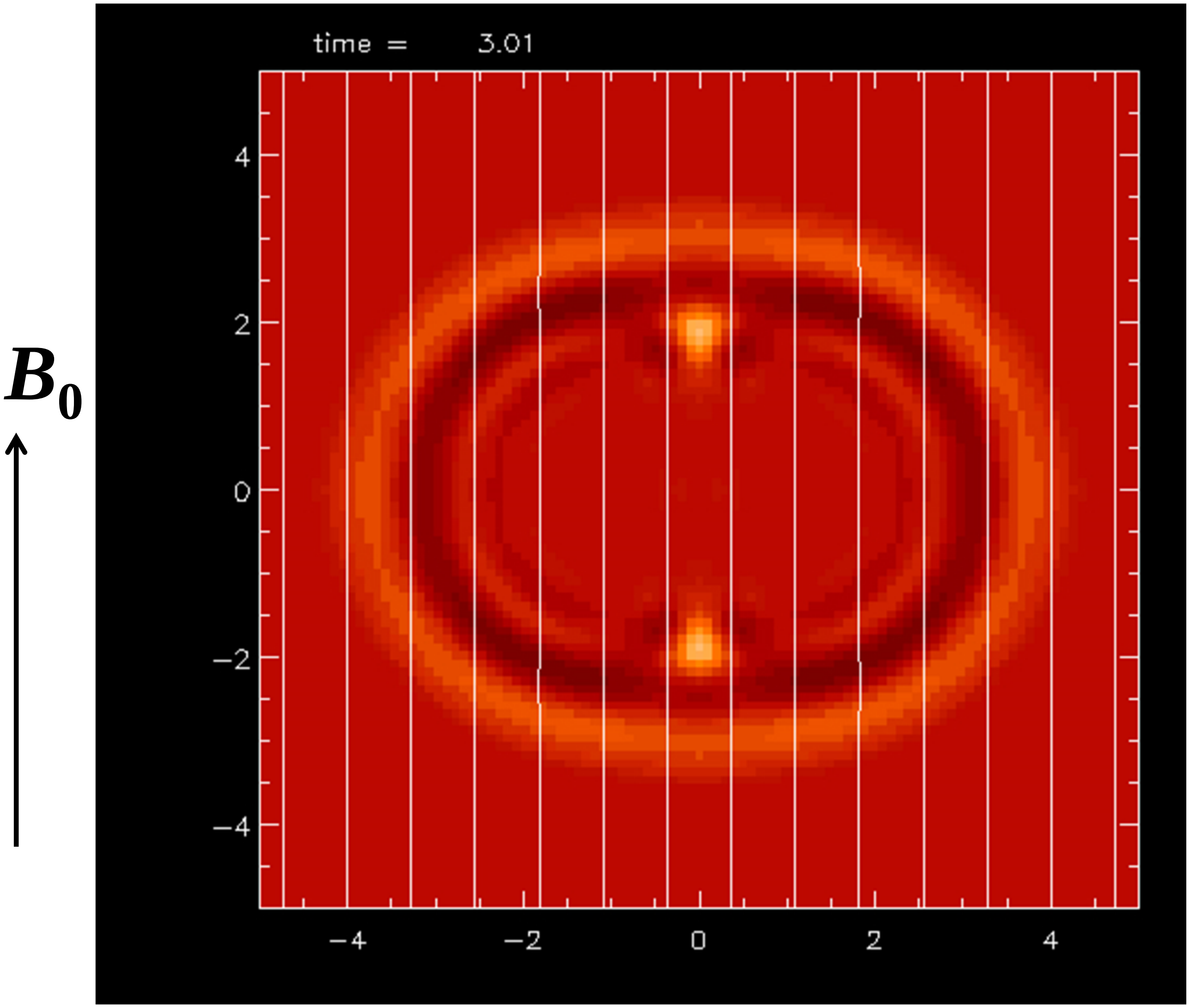
$$\frac{\gamma P}{B^2/4\pi} > 1, \text{ or}$$

$$P > P_M$$

the solution is a superposition of fast MHD waves (most pronounced across the field lines) and slow wave (mostly along the field lines) . This is the case of “high plasma beta”:

$$\beta = \frac{P}{P_M} = \frac{8\pi P}{B^2}$$

Weak magnetic field ($c_S > v_A$)



Pressure perturbation

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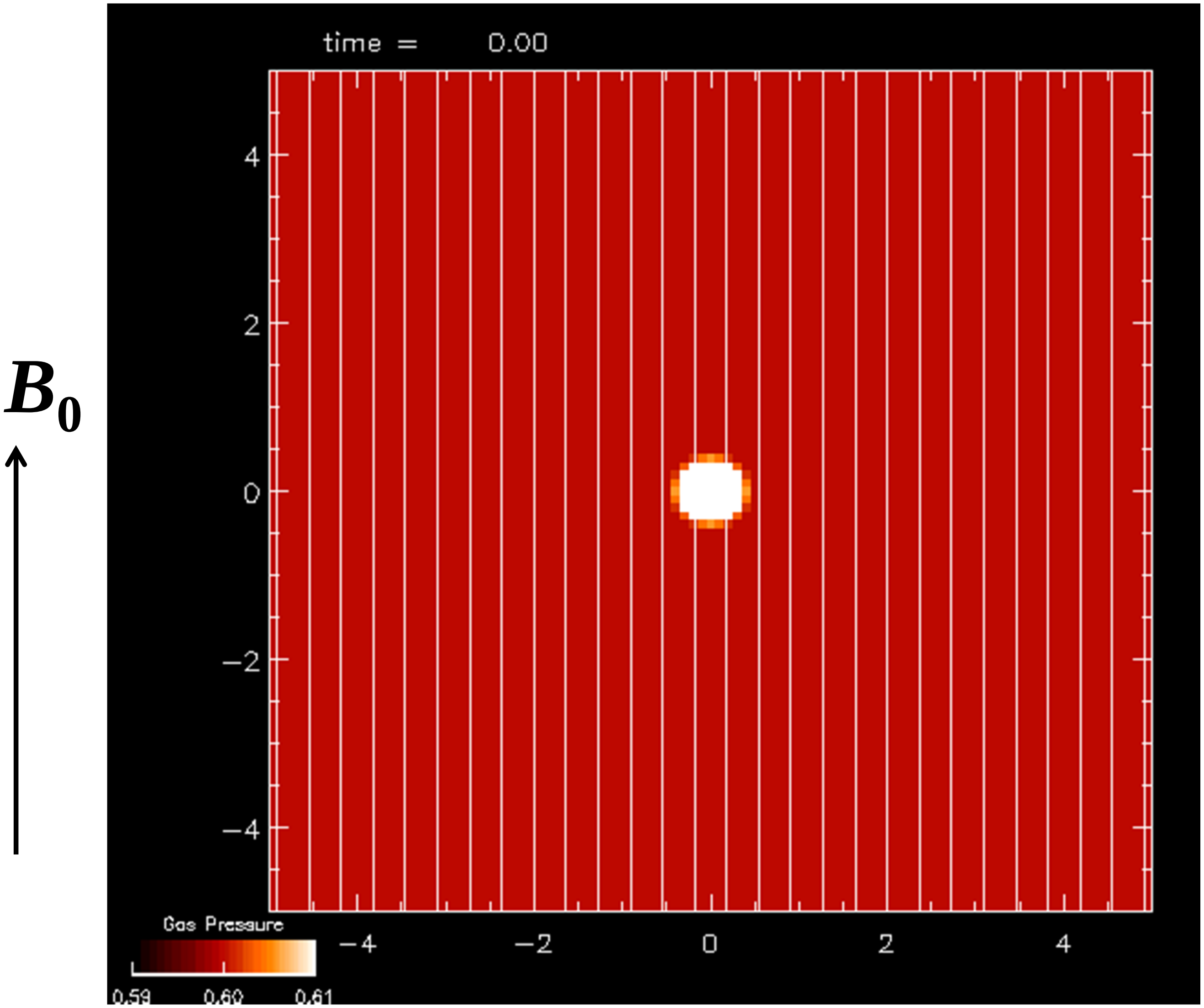
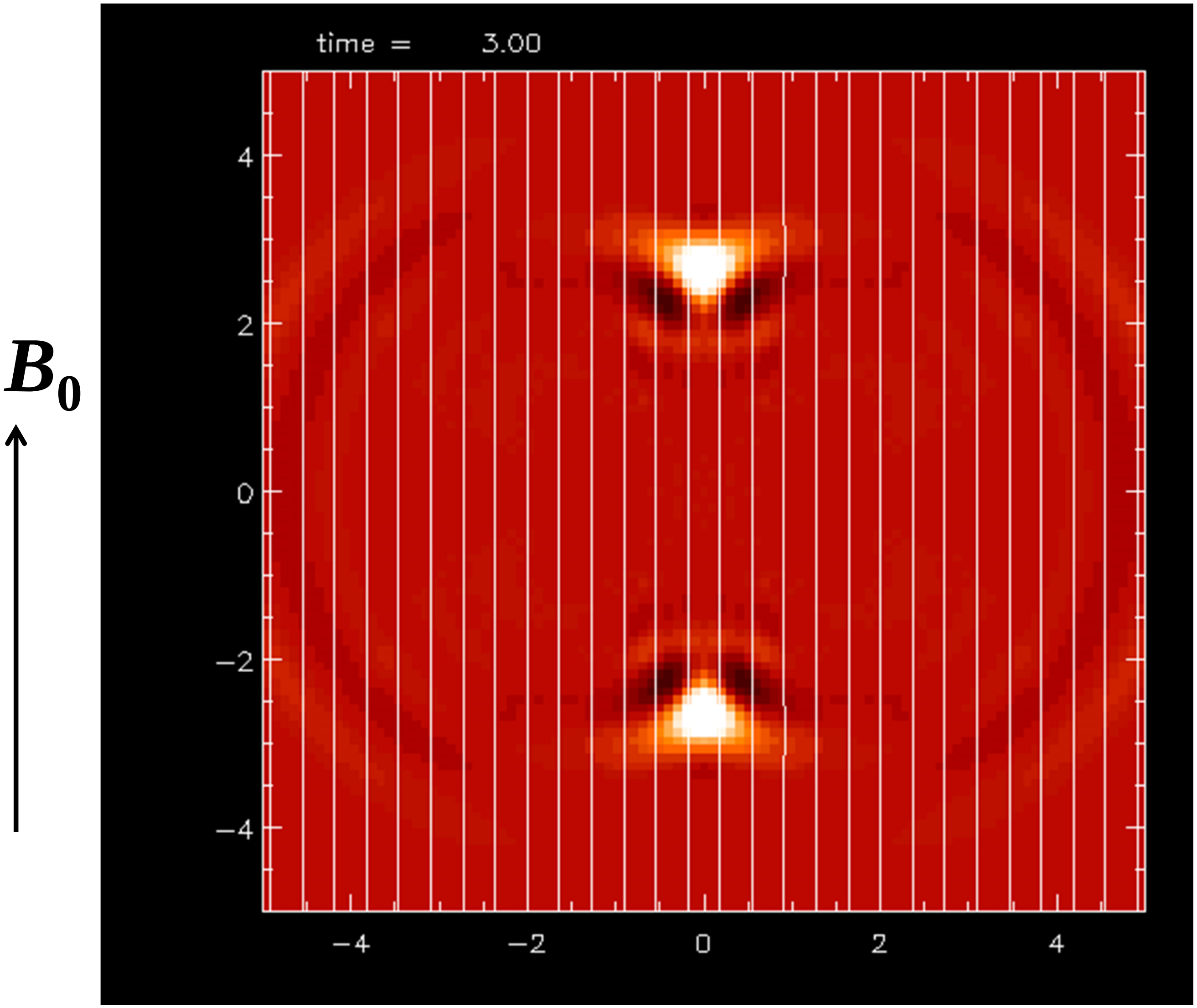
Strong magnetic field ($v_A > c_S$)



Pressure perturbation

In this case (“low plasma beta”) most of the energy is transported by the slow MHD waves along the magnetic field lines.

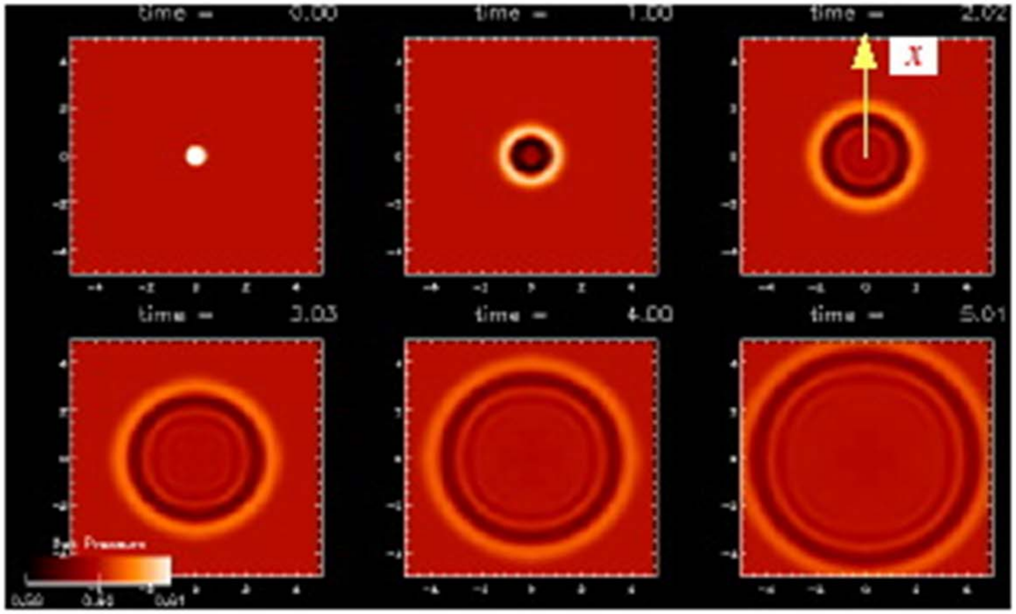
Strong magnetic field ($v_A > c_S$)



Pressure perturbation

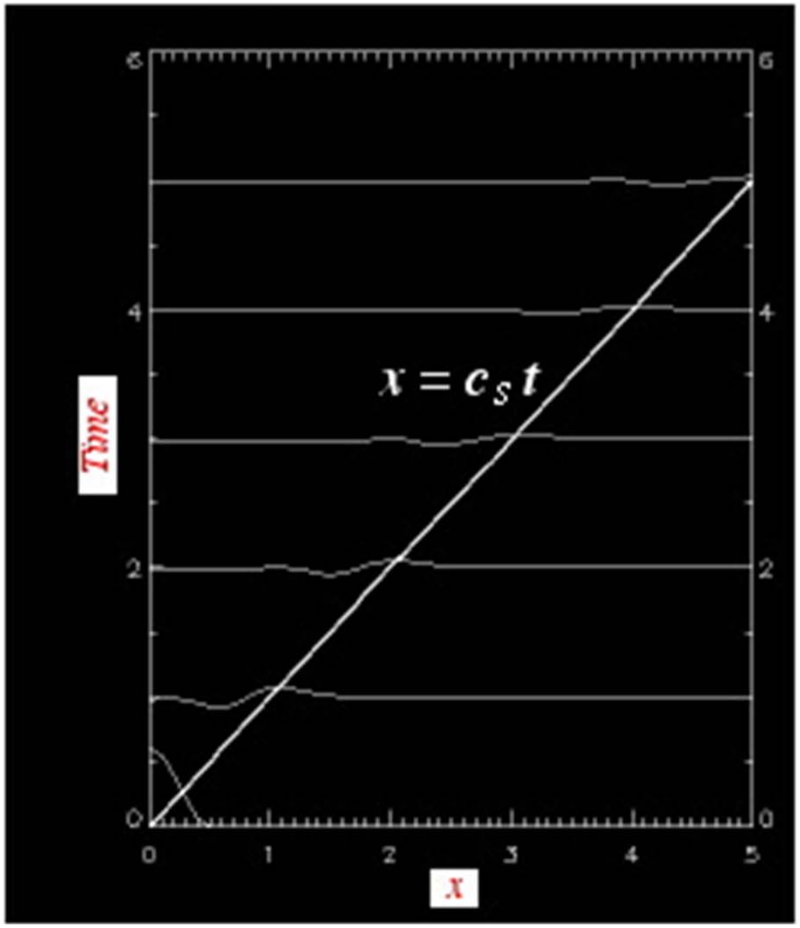
In this case (“low plasma beta”) most of the energy is transported by the slow MHD waves along the magnetic field lines.

Propagation of pure acoustic wave

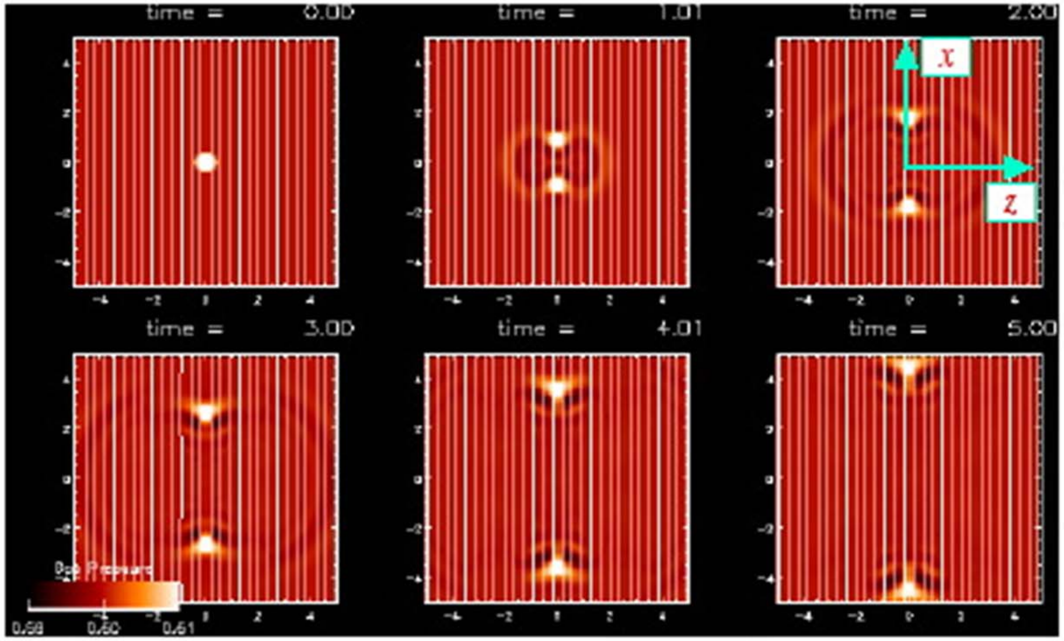


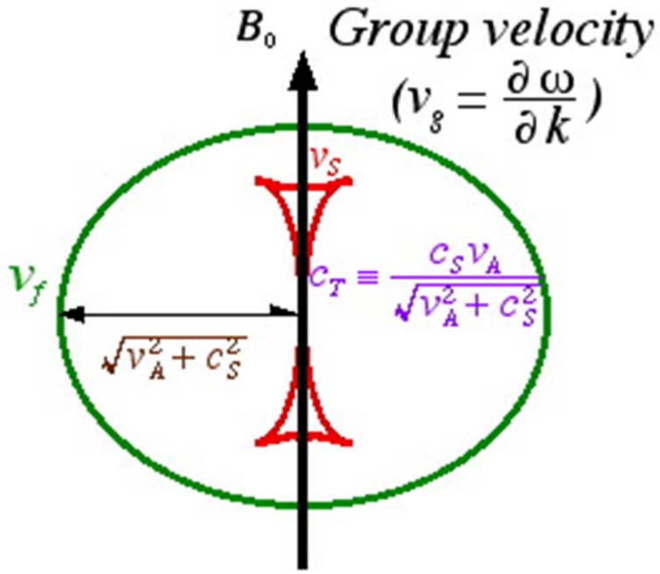
Time-distance diagram

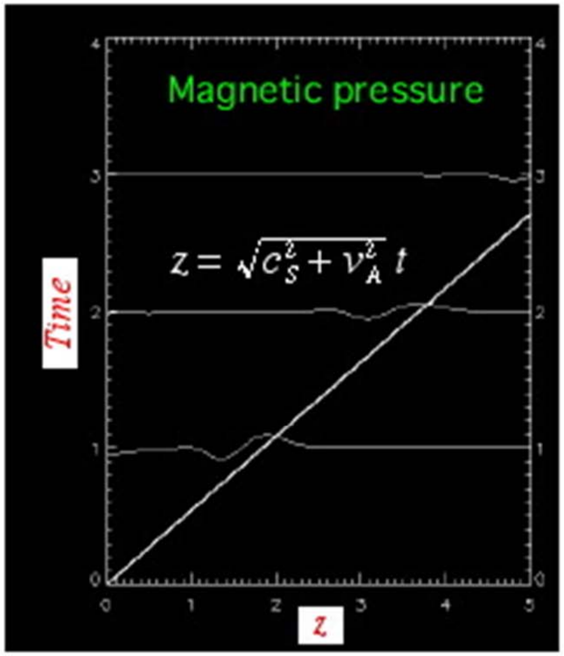
Propagation of sound wave



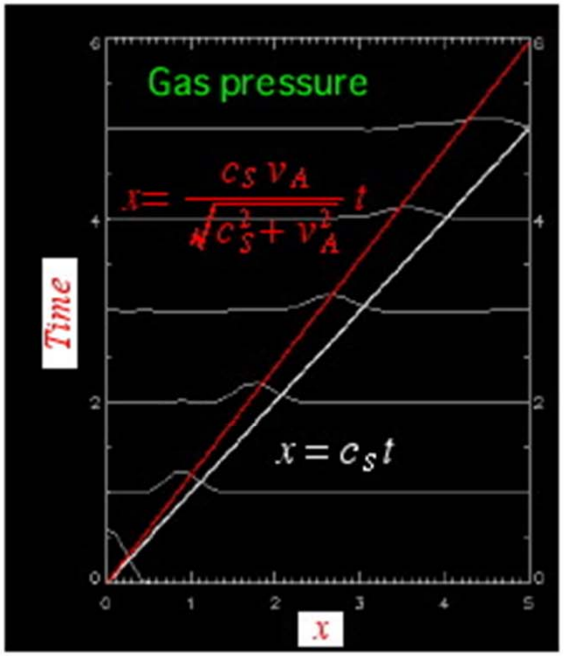
$$c_S = \sqrt{\gamma \frac{p_0}{\rho_0}} = 1$$







Propagation of fast mode wave (perpendicular to B_0)



Propagation of slow mode & tube mode waves (along B_0)

$$c_S = \sqrt{\gamma \frac{p_0}{\rho_0}} = 1$$

$$v_A = \frac{B_0}{\sqrt{4\pi\rho_0}} = \sqrt{\frac{2}{\gamma\beta}}\, c_S = 1.55$$

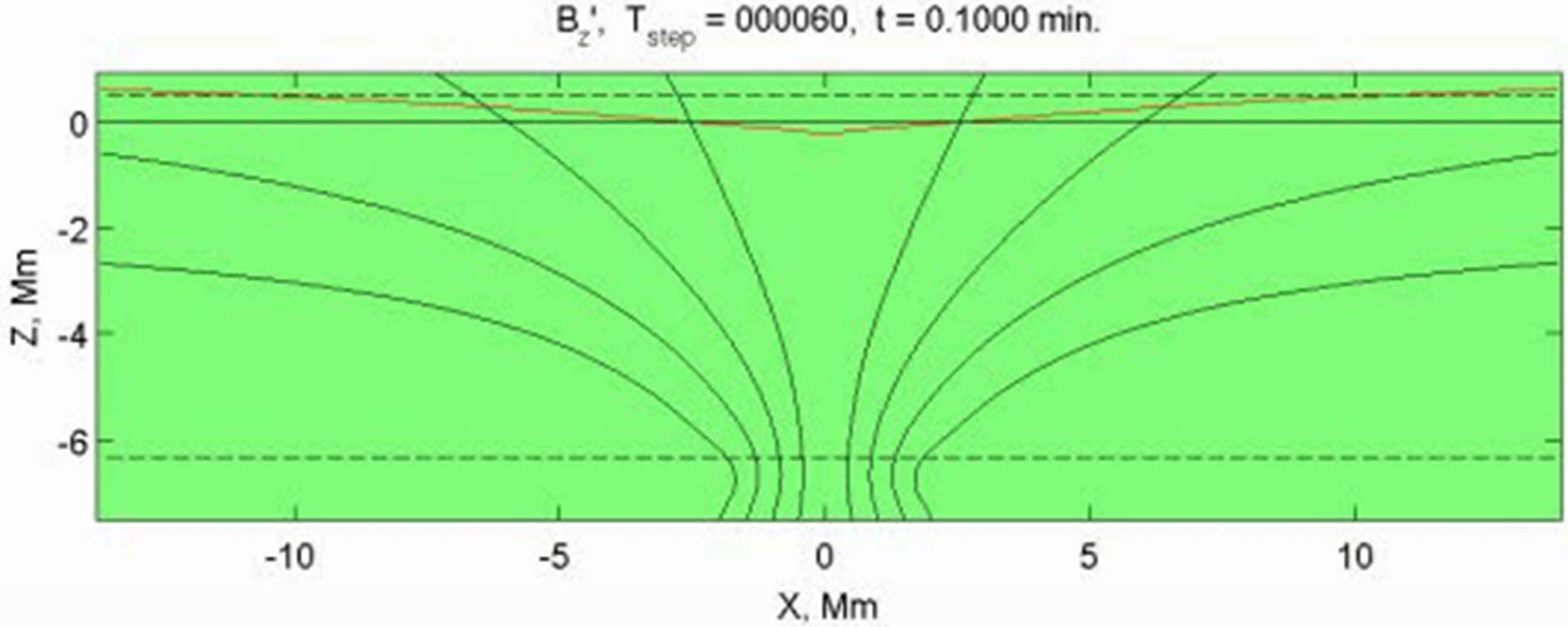
$$c_T = \frac{c_S v_A}{\sqrt{c_S^2 + v_A^2}} = 0.84$$

$$v_f = \sqrt{c_S^2 + v_A^2} = 1.84$$

Propagation of MHD waves: case $v_A > c_S$

Strong magnetic field transports wave energy preferentially along the magnetic field.

Numerical simulation of MHD wave propagation through a sunspot (K.Parchevsky)



Fast MHD waves travels across the magnetic field lines; slow MHD waves travels along the field lines into the deeper layers.