9. Theory of solar oscillations

Basic Equations

Basic assumptions:

1. linearity: $\vec{v}/c_s << 1$
2. adiabaticity: $dS/dt = 0$
3. spherical symmetry of the background
4. magnetic forces and Reynolds stresses are negligible

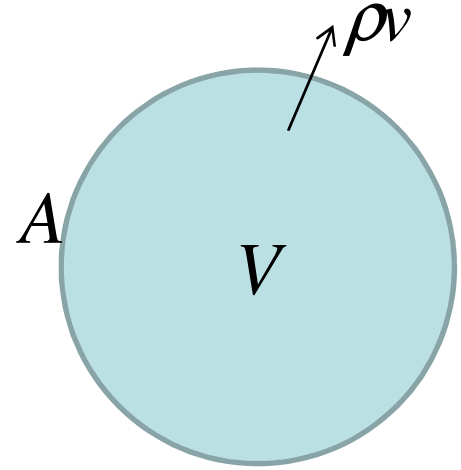
The basic equations are conservations of mass, momentum, energy and Newton's gravity law.

1. Conservation of mass (continuity equation):

The rate of mass change in a fluid element of volume V is equal to the mass flux through the surface of this element (of area A):

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_A \rho\vec{v}d\vec{a} = -\int_V \nabla(\rho\vec{v})dV.$$

divergence



Then,

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v}) = 0,$$

or

$$\frac{d\rho}{dt} + \rho\nabla\vec{v} = 0.$$

2. Momentum equation (conservation of momentum of a fluid element): $$\rho \frac{d\vec{v}}{dt} = -\nabla P + \rho \vec{g},$$

where P is pressure, $\vec{g}$ is the gravity acceleration, which can be expressed in terms of gravitational potential Φ: $\vec{g} = \nabla\Phi$.

Also, $\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$. This is the 'material' derivative.

$$e.g.\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$ for v_x component

3. Adiabaticity equation (conservation of energy) for a fluid element:

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0, \qquad \text{or} \qquad \frac{dP}{dt} = c^2 \frac{d\rho}{dt},$$

where $c^2 = \gamma P/\rho$ is the adiabatic sound speed.

4. Poisson equation: $$\nabla^2\Phi = 4\pi G\rho.$$

Plan to solve the solar oscillation equations

1. Linearize - consider small-amplitude oscillations.
2. Neglect the perturbations of the gravitational potential (Cowling approximation).
3. Write the linearized equations in the spherical coordinates: r, θ, ϕ.
4. Consider harmonic (periodic) oscillations
5. Separate the radial and angular coordinates.
6. Show that the angular dependence can be represented by spherical harmonics.
7. Derive equations for the radial dependence, representing the eigenvalue problem for the normal modes
8. Solve the eigenvalue problem in the asymptotic (short wave-length) JWKB approximation.
9. Investigate properties of p- and g-modes

1. Linearization

Consider small perturbations of a stationary spherically symmetrical star in the hydrostatic equilibrium:

$$v_0 = 0,\ \rho = \rho_0(r),\ P = P_0(r).$$

If $\vec{\xi}(t)$ is a vector of displacement of a fluid element then velocity of this element:

$$\vec{v} = \frac{d\xi}{dt} \approx \frac{\partial\vec{\xi}}{\partial t}.$$

Perturbations of scalar variables ρ, P, Φ are two types: **Eulerian, at a fixed position** $\vec{r}$:

$$\rho(\vec{r},t) = \rho_0(r) + \rho'(\vec{r},t),$$

and **Lagrangian perturbation in the moving element**:

$$\rho(\vec{r}+\vec{\xi}) = \rho_0(r) + \delta\rho(\vec{r},t).$$

The Eulerian and Lagrangian perturbations are related to each other:

$$\delta\rho = \rho' + (\vec{\xi}\cdot\nabla\rho_0) = \rho' + (\vec{\xi}\cdot\vec{e}_r)\frac{d\rho_0}{dr} = \rho' + \xi_r\frac{d\rho_0}{dr},$$

where $\vec{e}_r$ is a radial unit vector. In our case, the density gradient is radial.

Then, the linearized equations are:

$$\rho' + \nabla(\rho_0 \vec{\xi}) = 0,$$ the continuity (mass conservation) equation

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla P' - g_0 \vec{e}_r \rho' + \rho_0 \nabla \Phi',$$ the momentum equation

$$P' + \xi_r \frac{dP_0}{dr} = c_0^2 (\rho' + \xi_r \frac{d\rho_0}{dr}),$$ the adibaticity (energy) equation, or

$$\delta P = c_0^2 \delta \rho$$ for the Largangian perturbations of pressure and density.

$$\nabla^2 \Phi' = 4\pi G \rho'.$$ the equation for the gravitational potential

2. Cowling approximation: $\Phi' = 0.$

3. Consider the linearized equations in the spherical coordinates

$r, \theta, \phi:$ $\quad \vec{\xi} = \xi_r \vec{e}_r + \xi_\theta \vec{e}_\theta + \xi_\phi \vec{e}_\phi \equiv \xi_r \vec{e}_r + \vec{\xi}_h,$

where $\vec{\xi}_h = \xi_\theta \vec{e}_\theta + \xi_\phi \vec{e}_\phi$ is the horizontal component of displacement.

$$\nabla \vec{\xi} \equiv \mathrm{div}\,\xi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\xi_\theta) + \frac{1}{r\sin\theta}\frac{\partial \xi_\phi}{\partial\phi} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r}\nabla_h \vec{\xi}_h.$$

4. Consider periodic perturbations with frequency ω:

$$\vec{\xi} \propto e^{i\omega t} Y_l^m(\theta,\phi) = C P_l^m(\theta) e^{im\phi + i\omega t}$$

$\nu = \omega / 2\pi,$ where ν is the cyclic frequency (measured in Hz),

and ω is the angular frequency (measure in rad/s).

Then, in the Cowling approximation, we get (leaving out subscript 0 for unperturbed variables):

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho \xi_r) + \frac{\rho}{r}\nabla_h \vec{\xi}_h = 0,$$ the continuity equation

$$-\omega^2 \rho \xi_r = -\frac{\partial P'}{\partial r} + g\rho',$$ the radial component of the momentum equation

$$-\omega^2 \rho \vec{\xi}_h = -\frac{1}{r}\nabla_h P',$$ the horizontal component of the momentum equation

$$\rho' = \frac{1}{c^2}P' + \frac{\rho N^2}{g}\xi_r,$$ the adiabatic equation

where $N^2 = g\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right)$ is the Brunt-Vaisala frequency.

Boundary conditions:

$\xi_r(r=0) = 0,$ - displacement at the Sun's center is zero,

(or a regularity condition for $l = 1$).

$\delta P(r = R) = 0,$ - Lagrangian pressure perturbation at the solar surface is zero.

(this is equivalent to absence of external forces).

Also, we assume that the solution is regular at the poles $\theta = 0, \pi$.

5. Consider the separation of radial and angular variables in the form:

$$\rho'(r,\theta,\phi) = \rho'(r)\cdot f(\theta,\phi),$$
$$P'(r,\theta,\phi) = P'(r)\cdot f(\theta,\phi),$$
$$\xi_r(r,\theta,\phi) = \xi_r(r)\cdot f(\theta,\phi),$$
$$\vec{\xi}_h(r,\theta,\phi) = \xi_h(r)\nabla_h f(\theta,\phi).$$

Then, the continuity equation is:

$$\left[\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\xi_r)\right] f(\theta,\phi) + \frac{\rho}{r}\xi_h\nabla_h^2 f = 0.$$

The variables are separated if

$$\nabla_h^2 f = \alpha f,$$

where α is a constant.

This equation has non-zero solutions regular at the poles, $\theta = 0,\pi$ only when

$$\alpha = -l(l+1),$$

where l is an integer.

6. The non-zero solution of equation $\nabla_h^2 f + l(l+1)f = 0$ represents the spherical harmonics:

$$f(\theta,\phi) = Y_l^m(\theta,\phi) = CP_l^m(\theta)e^{im\phi},$$

where $P_l^m(\theta)$ is the Legendre function.

7. Derive equations for the radial dependence, representing the eigenvalue problem for the normal modes

After the separation of variables the continuity equation for the radial dependence $\rho'(r)$ is

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho \xi_r\right) - \frac{l(l+1)}{r^2}\rho\xi_h = 0.$$

Compare with the original equation: $\rho' + \nabla(\rho_0 \vec{\xi}) = 0,$

and with this equation in the spherical coordinates:

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho \xi_r) + \frac{\rho}{r}\nabla_h \vec{\xi}_h = 0,$$

Transform this equation in terms of 2 variables: ξ_r and P'
- radial displacement and Eulerian pressure perturbation.

The horizontal component of displacement ξ_h can be determined from the horizontal component of the momentum equation:

$$-\omega^2 \rho \xi_h(r) = -\frac{1}{r} P'(r),$$

or

$$\xi_h = \frac{1}{\omega^2 \rho r} P'.$$

Substituting this into the continuity equation we get:

$$\rho \frac{d\xi_r}{dr} + \xi_h \frac{d\rho}{dr} + \frac{2}{r}\rho\xi_r + \frac{P'}{c^2} + \frac{\rho N^2}{g}\xi_r - \frac{L^2}{r^2\omega^2\rho}P' = 0,$$

where we define $L^2 = l(l+1)$ (note the similarity to quantum mechanics).

Using the hydrostatic equation for the background (unperturbed) state

$$\frac{dP}{dr} = -g\rho,$$

finally get:

$$\frac{d\xi_r}{dr} + \frac{2}{r}\xi_r - \frac{g}{c^2}\xi_r + \left(1 - \frac{L^2 c^2}{r^2\omega^2}\right)\frac{P'}{\rho c^2} = 0,$$

or

$$\frac{d\xi_r}{dr} + \frac{2}{r}\xi_r - \frac{g}{c^2}\xi_r + \left(1 - \frac{S_l^2}{\omega^2}\right)\frac{P'}{\rho c^2} = 0,$$

where $S_l^2 = \frac{L^2 c^2}{r^2}$ is **the Lamb frequency,** $L^2 = l(l+1)$, $c^2(r) = \gamma P/\rho$ is the squared sound speed, $g(r) = Gm(r)/r^2$ is the gravity acceleration at radius r.

Similarly, the momentum equation is:

$$\frac{dP'}{dr} + \frac{g}{c^2} P' + (N^2 - \omega^2)\rho\xi_r = 0,$$

where N^2 is the Brunt-Vaisala frequency.

The lower boundary condition: $\xi_r = 0,$ (or a regularity condition).

The upper boundary condition: $\delta P = P' + \frac{dP}{dr}\xi_r = 0,$

or using the hydrostatic equation: $P' - g\rho\xi_r = 0.$

From the horizontal component of the momentum equation:

$$P' = \omega^2 \rho r \xi_h,$$

Then from the upper boundary condition: $\frac{\xi_h}{\xi_r} = \frac{g}{\omega^2 r},$

that is the ratio of the horizontal and radial components of displacement is inverse proportional to squared frequency. However, this relation does not hold in observations, presumably, because of the external force caused by the solar atmosphere.

7. The derived equations with the boundary conditions constitute an eigenvalue problem for solar oscillation modes

$$\frac{d\xi_r}{dr} + \frac{2}{r}\xi_r - \frac{g}{c^2}\xi_r + \left(1 - \frac{S_l^2}{\omega^2}\right)\frac{P'}{\rho c^2} = 0,$$

ξ_r is the radial displacement
P' is the Eulerian pressure perturbation

$$\frac{dP'}{dr} + \frac{g}{c^2}P' + (N^2 - \omega^2)\rho\xi_r = 0.$$

Properties of oscillations depend on the signs of these coefficients in brackets.

$$S_l^2 = \frac{L^2 c^2}{r^2}$$ is the Lamb frequency. $L^2 = l(l+1)$

$$N^2 = g\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right)$$ is the Brunt-Vaisala frequency.

The lower boundary condition: $\xi_r = 0$, (or a regularity condition).

The upper boundary condition: $\delta P = P' + \frac{dP}{dr}\xi_r = 0,$

$$\frac{d\xi_r}{dr} + \frac{2}{r}\xi_r - \frac{g}{c^2}\xi_r + \left(1 - \frac{S_l^2}{\omega^2}\right)\frac{P'}{\rho c^2} = 0,$$

ξ_r is the radial displacement
P' is the Eulerian pressure perturbation

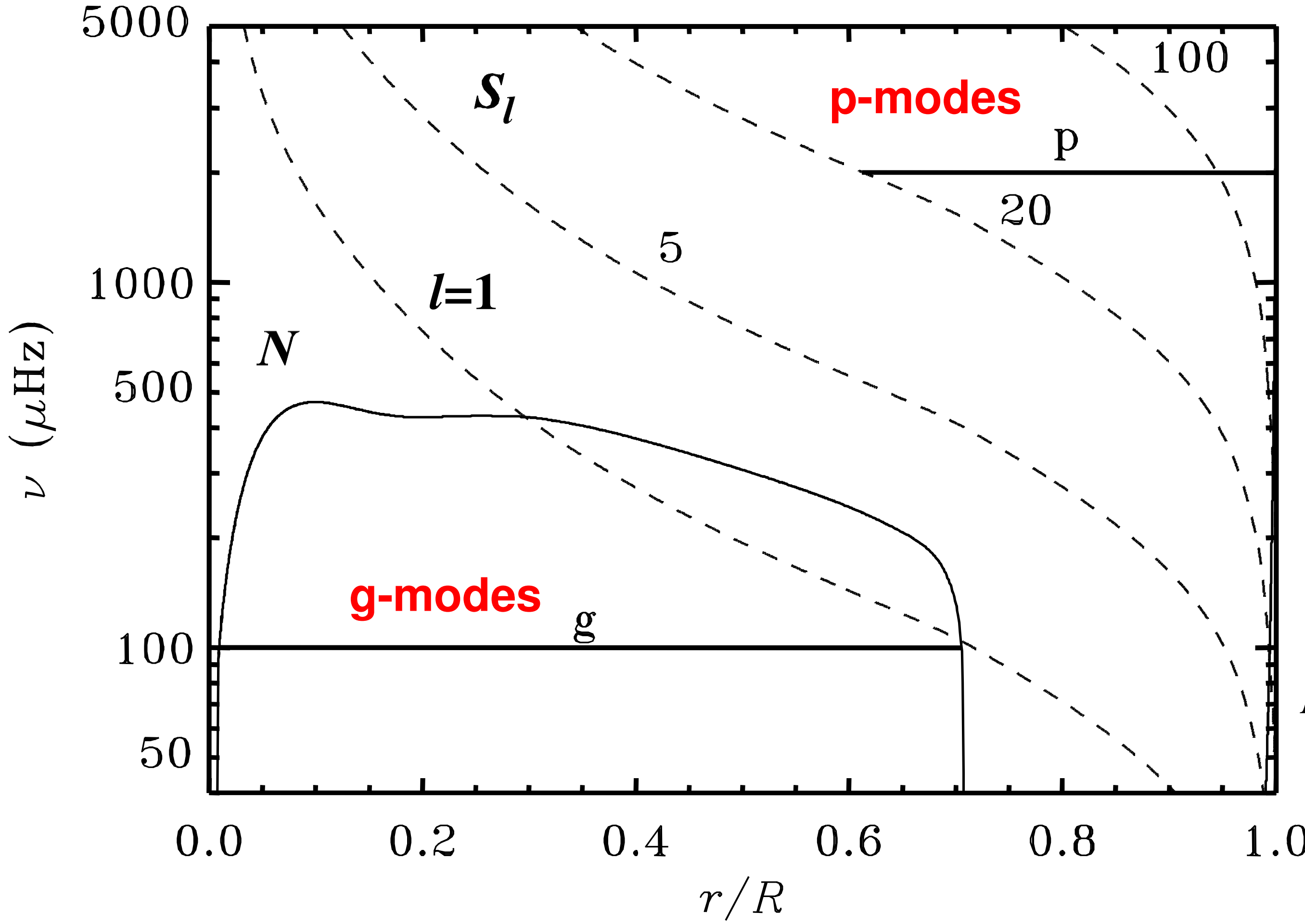
$$\frac{dP'}{dr} + \frac{g}{c^2}P' + (N^2 - \omega^2)\rho\xi_r = 0.$$

$$\frac{d\xi_r}{dr} + \left(1 - \frac{S_l^2}{\omega^2}\right)\frac{P'}{\rho c^2} \approx 0, \qquad \frac{dP'}{dr} + (N^2 - \omega^2)\rho\xi_r \approx 0.$$

$$\frac{d^2\xi_r}{dr^2} + \frac{1}{c^2\omega^2}\left(\omega^2 - S_l^2\right)(\omega^2 - N^2)\xi_r \approx 0,$$

solution is oscillatory if $\left(\omega^2 - S_l^2\right)(\omega^2 - N^2) > 0$

Propagation diagram of solar oscillations



p-modes (acoustic modes):

$$\omega > S_l \quad \omega > N$$

g-modes (internal gravity modes):

$$\omega < S_l \quad \omega < N$$

$$S_l^2 = \frac{L^2 c^2}{r^2}$$

the Lamb frequency.

$$N^2 = g\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right)$$

the Brunt-Vaisala frequency.

Buoyancy (Brunt-Vaisala) frequency N, and Lamb frequency S_l for $l = 1, 5, 20$ and 100 vs. fractional radius r/R for a standard solar model. The horizontal lines indicate the trapping regions for a g mode with frequency $\nu = 100\mu$ Hz, and for a p mode of degree $l = 20$ and $\nu = 2000\mu$ Hz.

8. JWKB (Jeffreys-Wentzel–Kramers–Brillouin) Solution (short-wavelength asymptotic approximation – similar to quantum mechanics)

We assume that only density $\rho(r)$ varies significantly among the solar properties in the oscillation equations, and seek for an oscillatory solution in the JWKB form:

$$\xi_r = A\rho^{-1/2} e^{ik_r r},$$

$$P' = B\rho^{1/2} e^{ik_r r},$$

where the radial wavenumber k_r is a slowly varying function of r.

Then,

$$\frac{d\xi_r}{dr} = A\rho^{-1/2}\left(ik_r + \frac{1}{2H}\right)e^{ik_r r},$$

$$\frac{dP'}{dr} = B\rho^{1/2}\left(ik_r - \frac{1}{2H}\right)e^{ik_r r},$$

where $H = -\left(\frac{d\log\rho}{dr}\right)^{-1}$ **is the density scale height**.

From the oscillation equations we get a linear system:

$$\left(ik_r + \frac{1}{2H}\right)A - \frac{g}{c^2}A + \frac{1}{c^2}\left(1 - \frac{S_l^2}{\omega^2}\right)B = 0,$$

$$\left(ik_r - \frac{1}{2H}\right)B + \frac{g}{c^2}B + (N^2 - \omega^2)A = 0.$$

The determinant of this system is equal zero when

$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} + \frac{S_l^2}{c^2\omega^2}\left(N^2 - \omega^2\right)$$

where $\omega_c = \frac{c}{2H}$ is the **acoustic cut-off frequency**

(use the relation: $N^2 = g/H - g^2/c^2$).

The solar waves propagate in the regions where $k_r^2 > 0$.

If $k_r^2 < 0$, the waves exponentially decay ('evanescent').

Properties of Solar Oscillation Modes

Equation $$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} + \frac{S_l^2}{c^2\omega^2}\left(N^2 - \omega^2\right)$$

represents a **dispersion relation of solar waves**.

It relates frequency ω with radial wavenumber k_r and angular order l.

Consider two simple cases:

1: the high-frequency case. If $\omega^2 >> N^2$ then

$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{S_l^2}{c^2}$$

or $$\omega^2 = \omega_c^2 + k_r^2 c^2 + k_h^2 c^2,$$

where $k_h = S_l / c \equiv \frac{L}{r} \equiv \frac{\sqrt{l(l+1)}}{r}$ is **the horizontal wave number**.

Then, $k^2 = k_r^2 + k_h^2$ is the squared total wavenumber.

Finally, $\omega^2 = \omega_c^2 + k^2 c^2$, where $\omega_c = \frac{c}{2H}$ is the acoustic cut-off frequency.

This is **the dispersion relation for acoustic (p) modes**; ω_c **is the acoustic cutoff frequency. Physically, the waves with frequencies below the acoustic cutoff frequency cannot propagate. Their wavelength becomes shorter than the density scale height. For the Sun** $\nu_c \equiv \omega_c / 2\pi \approx 5\ mHz.$ **(c~10 km/s, H~150km).**

$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} + \frac{S_l^2}{c^2\omega^2}\left(N^2 - \omega^2\right)$$

<u>2: consider the low-frequency case when</u> $\omega^2 << S_l^2$

then $k_r^2 = \frac{S_l^2}{c^2\omega^2}(N^2 - \omega^2)$ (remember $S_l = ck_h = cL/r$)

Then, $\omega^2 = \frac{k_h^2 N^2}{k^2} \equiv N^2 \cos^2\theta,$ where $k^2 = k_r^2 + k_h^2$

where θ is the angle between wavevector k and the horizontal direction.

This is a dispersion relation for internal gravity (g). modes. They propagate mostly horizontally.

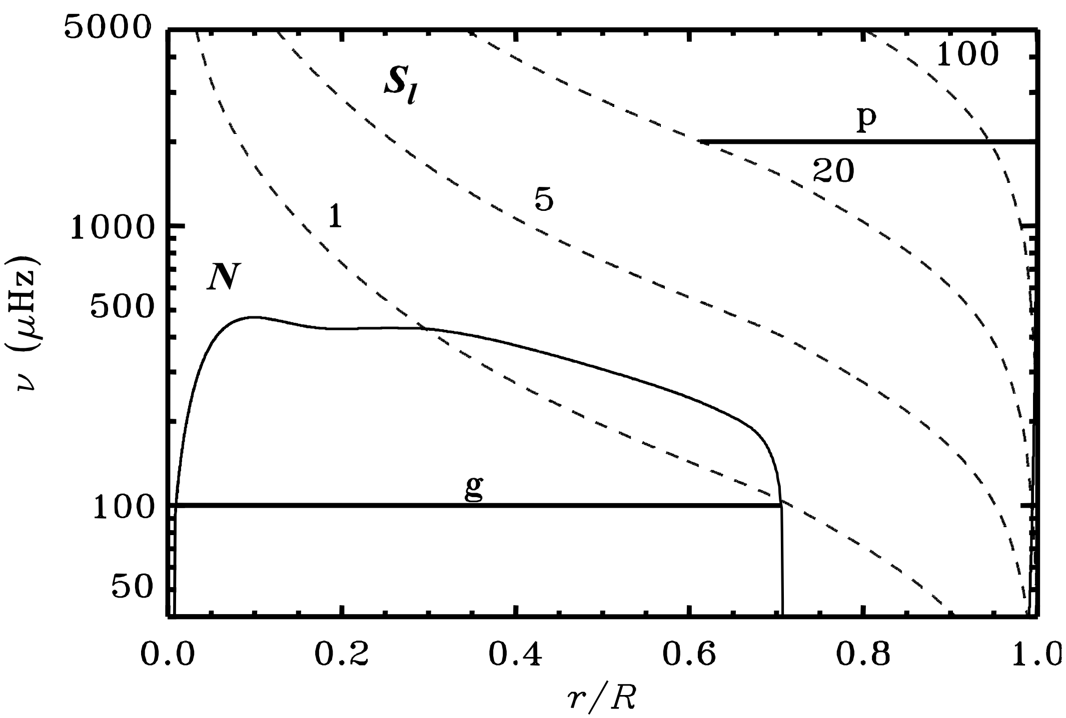
Normal modes of solar oscillations

The frequencies of normal modes are determined for the Borh quantization rule (resonant condition):

$$\int_{r_1}^{r_2} k_r dr = \pi(n+\alpha),$$

where r_1 and r_2 are the radii of the turning points where k_r =0, n is a radial order -integer number, and α is a phase shift which depends on properties of the reflecting boundaries.

$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} + \frac{S_l^2}{c^2\omega^2}\left(N^2 - \omega^2\right)$$



$c(r)$ is the sound speed

$$\omega_c = \frac{c}{2H}$$

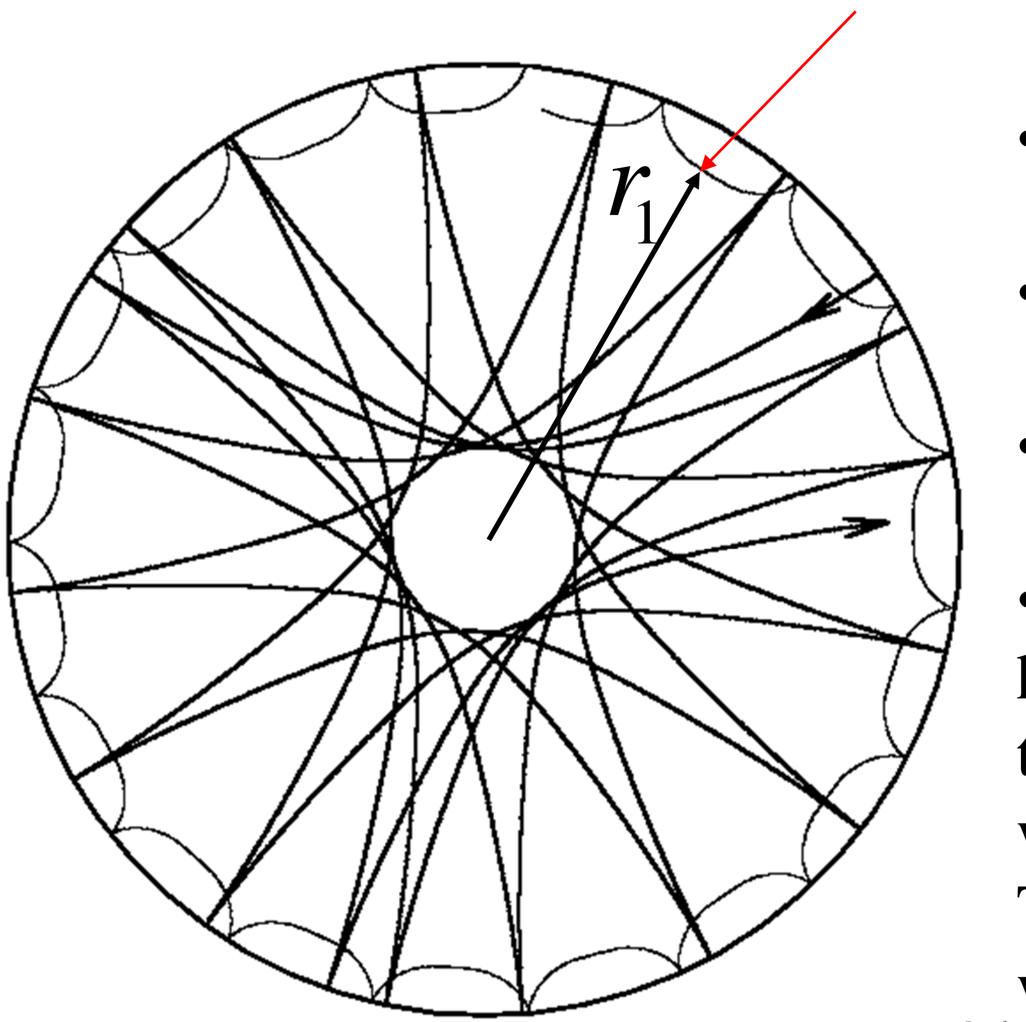
is the acoustic cut-off frequency; it has very sharp increase at r/R=1

$$H = \left(\frac{d\log\rho}{dr}\right)^{-1},$$

$$S_l^2 = \frac{L^2c^2}{r^2} \qquad L^2 = l(l+1)$$

$$N^2 = g/H - g^2/c^2$$

P-mode ray paths



$$k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{S_l^2}{c^2}$$

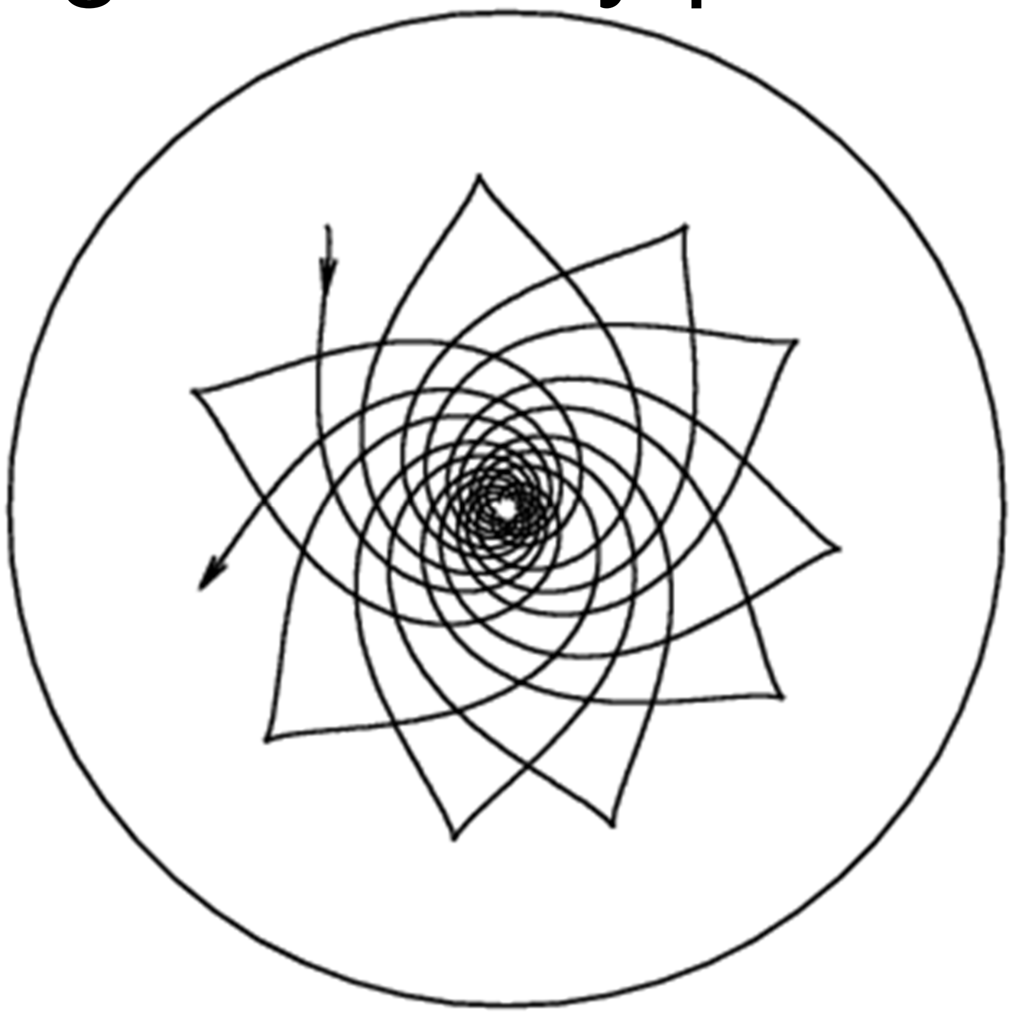
- The waves propagate where $k_r^2 > 0$.
- The waves are evanescent where $k_r^2 < 0$
- The wave turning points are located where $k_r^2 = 0$.
- Because $\omega_c = c/2H$

has a sharp peak near the surface the upper turning point (r_2) is where $\omega = \omega_c$

The lower turning point (r_1) is where $\omega = S_l = (L/r)c = k_h c$

· where the horizontal phase speed $\omega / k_h = c$ is equal to the sound speed.

g-mode ray paths



g-modes propagate only in the radiative zone which is convectively stable $N^2 > 0$

Calculation of normal mode frequencies

Estimate frequencies of normal modes for these 2 cases.

1. **p-modes:**

propagating region: $k_r^2 > 0$

turning points $k_r^2 = 0$: $$\omega^2 = \omega_c^2 + \frac{L^2 c^2}{r^2}.$$

For the lower turning point in the interior: $\omega_c << \omega$.

Then, $\omega \approx \frac{Lc}{r}$, or $\frac{c(r_1)}{r_1} = \frac{\omega}{L}$ is the equation for the lower turning point.

The upper turning point: $\omega_c(r_2) \approx \omega$. Since $\omega_c(r)$ is a steep function of r near the surface, $r_2 \approx R$.

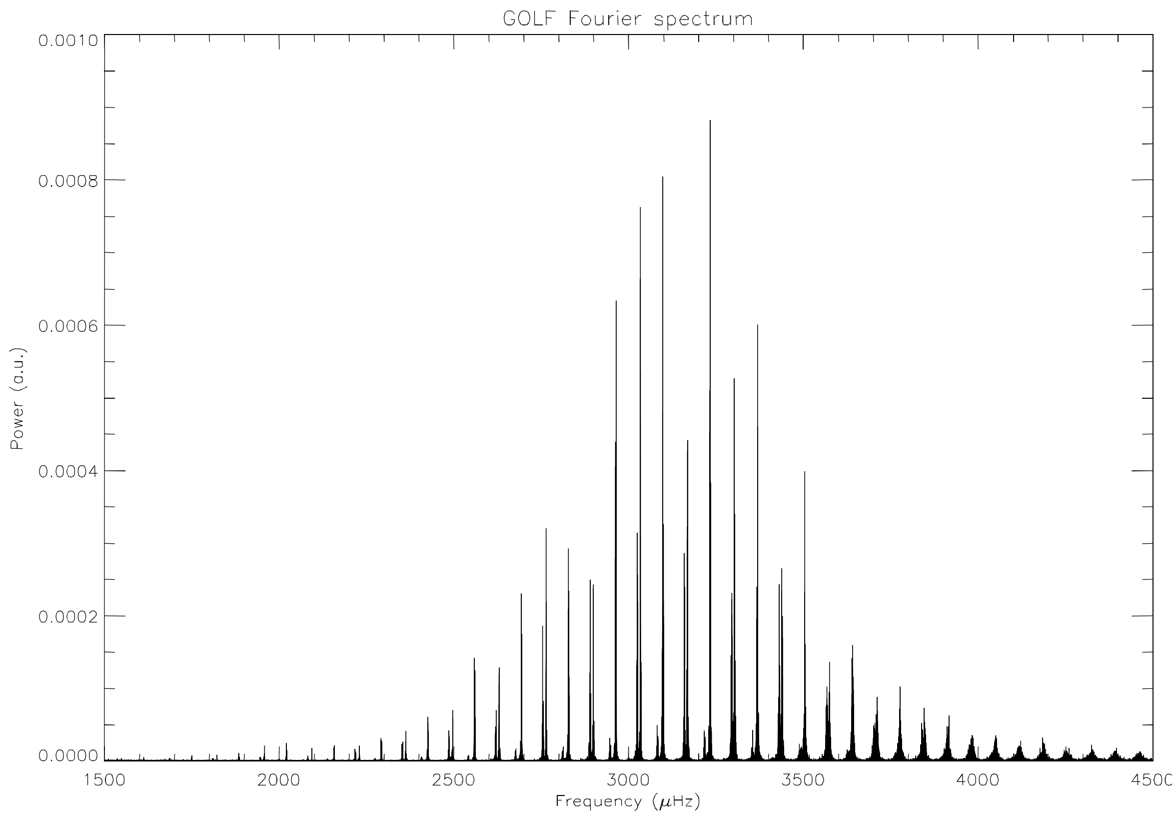
Then, the resonant condition for p-modes is: $$\int_{r_1}^{R} \sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}\, dr = \pi(n + \alpha)$$

Abel integral equation.

Low-degree p-modes

For $l << n$, $r_1 \approx 0$, and we get: $$\omega \approx \frac{\pi(n + L/2 + \alpha)}{\int_0^R \frac{dr}{c}}.$$

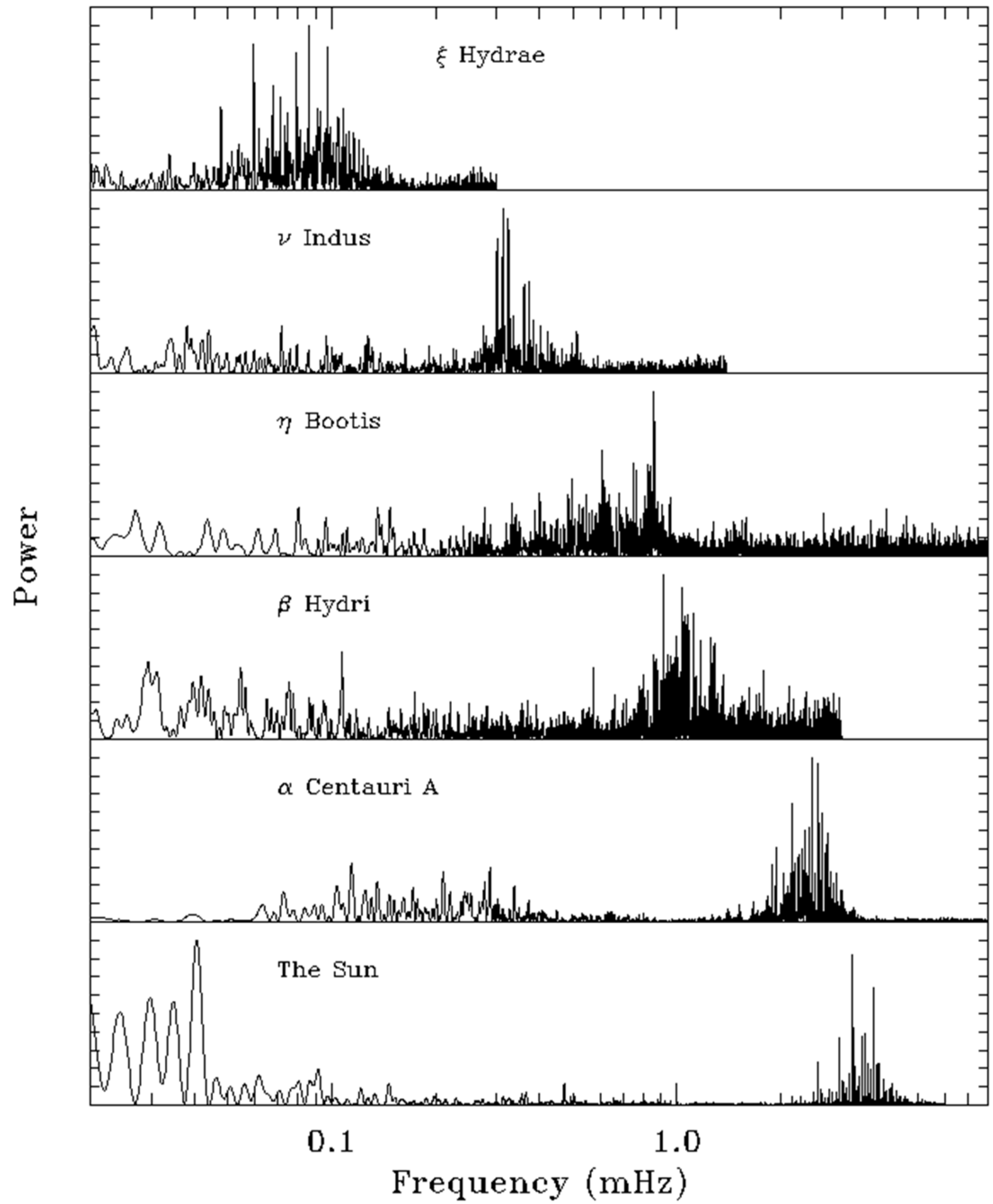
That is the spectrum of low-degree p-modes is approximately equidistant with frequency spacing: $$\Delta\nu = \left(4\int_0^R \frac{dr}{c}\right)^{-1}.$$



Maximum amplitude is around 3,300 μHz, or 3.3 mHz. The corresponding oscillation period is 300 seconds or 5 minutes.

Asteroseismology

Bedding & Kjeldsen (2003)



Frequencies of g-modes:

The turning points are determined from equation: $N(r) = \omega.$

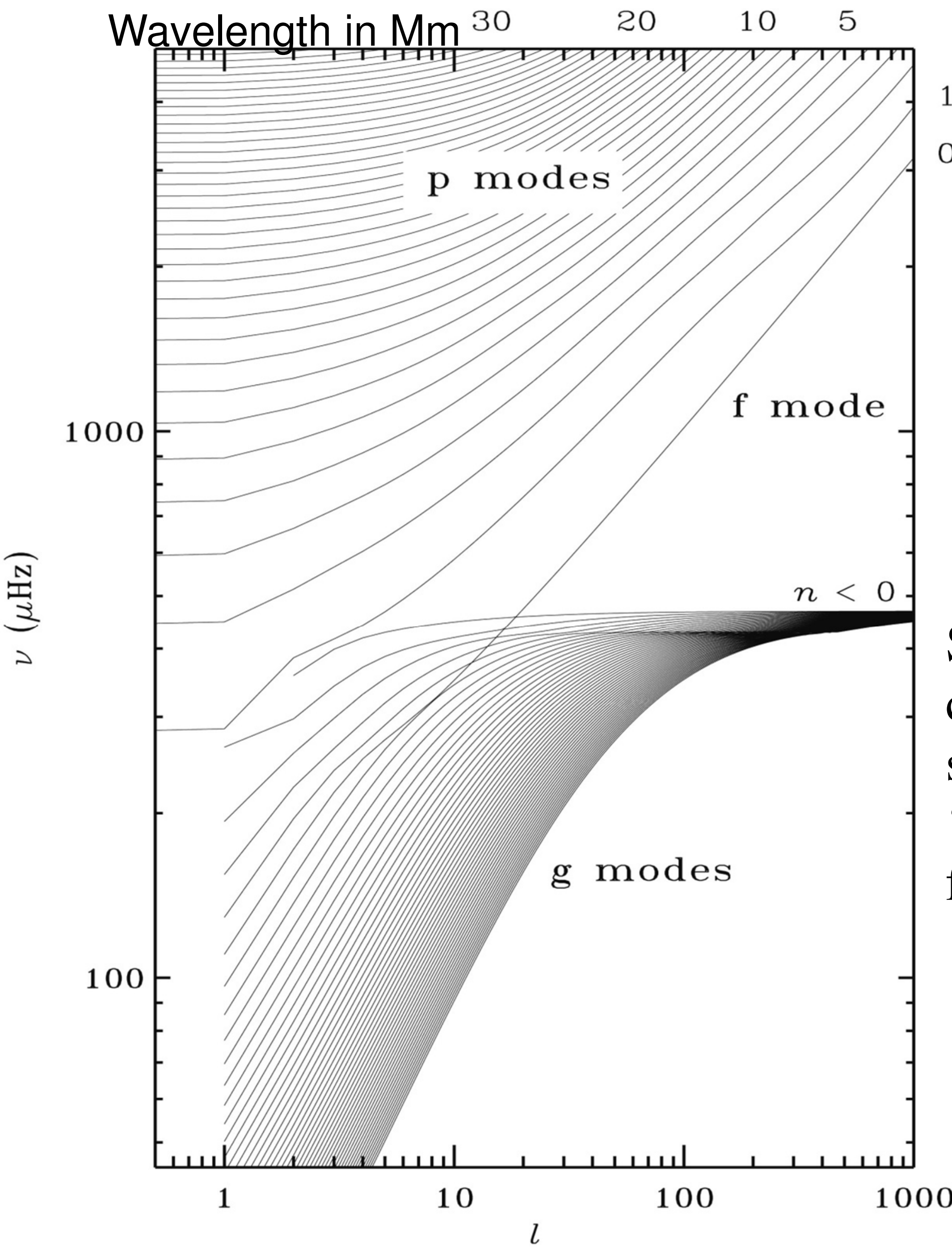
In the propagation region, $k_r > 0$, far from the turning points ($N >> \omega$):

$$k_r \approx \frac{LN}{r\omega}.$$

Then, from the resonant condition:

$$\int_{r_1}^{r_2} \frac{L}{\omega} N \frac{dr}{r} = \pi(n+\alpha).$$

we find: $$\omega \approx \frac{L\int_{r_1}^{r_2} N \frac{dr}{r}}{\pi(n+\alpha)}.$$



Spectrum of normal modes calculated for a standard solar model. Note the 'avoided crossing effect' for f and g-modes.

Surface gravity waves (f-mode)

These wave propagate at the surface boundary where Lagrangian pressure perturbation $\delta P \sim 0$.

Consider the oscillation equations in terms of δP by making use of the relation between Eulerian and Lagrangian variables: $P' = \delta P + g\rho\xi_r$.

$$\frac{d\xi_r}{dr} - \frac{L^2 g}{\omega^2 r^2}\xi_r + \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)\frac{\delta P}{\rho c^2} = 0,$$

$$\frac{d\delta P}{dr} + \frac{L^2 g}{\omega^2 r^2}\delta P - \frac{g\rho f}{r}\xi_r = 0,$$

where $$f \approx \frac{\omega^2 r}{g} - \frac{L^2 g}{\omega^2 r}.$$

These equations have a peculiar solution: $\delta P = 0, \; f = 0.$

For this solution: $$\omega^2 = \frac{Lg}{R} = k_h g$$

-dispersion relation for f-mode.

The eigenfunction equation: $$\frac{d\xi_r}{dr} - \frac{L}{r}\xi_r = 0$$

has a solution $\xi_r \propto e^{k_h(r-R)}$ exponentially decaying with depth.