Physics 780. Problem set 2

October 2, 2018
Due: Tuesday, October 16, 2018

Problem 2.1. Consider a uniform cloud of gas hydrogen of radius R, assumed neutral and atomic with number density n, and temperature T. Estimate (to within a factor of order unity) the thermal energy U and the magnitude Ω of the gravitational energy. The Jeans' criterion for collapse is, approximately, that $\Omega > U$.

(a) Typical conditions for a cool cloud of interstellar gas correspond to $n = 10^2$ cm^{-3} and $T = 10^2$ K. Calculate the minimum mass of gas required for collapse.

(b) How does this compare with the mass of a typical star? What does this tell you about the process of star formation?

(c) As the collapse proceeds, do conditions become more or less favorable for collapse?

Problem 2.2. Homology scaling laws for main-sequence stars provide a useful method of determining how gross observable parameters such as luminosity L, radius R, and effective temperature T_{eff} depend on mass M and composition (hydrogen abundance X and heavy element abundance Z). The scaling laws are obtained by seeking homologous transformations of a solution to the structure equations. For any dependent variable q, say, one sets

$$q(r) = Q\tilde{q}(x),$$

where $x = r/R$, and demands that the function $\tilde{q}(x)$ is independent of the scaling factor Q.

(a) Using the scaling laws:

$$R \propto Z^{0.15} X^{0.68} M^{0.077} \tag{1}$$

$$L \propto Z^{-1.1} X^{-5.0} M^{5.46} \tag{2}$$

and the black-body radiation law obtain the relation between L, T_{eff} and composition.

(b) Estimate the slope of the main sequence on the observed Hertzsprung-Russel diagram ($\log L$ vs $\log T_{\rm eff}$) and compare with the scaling law.

(c) Can you predict how the position of the Sun on the H-R diagram will change with time (evolutionary track of the Sun)?

(d) (extra credit) Verify Eqs. (1) and (2) using the equations of stellar structure and the power law approximations to opacity $\kappa \propto Z X^{0.44} \rho T^{-3.5}$, molecular weight $\mu \propto X^{-0.57}$ and energy-generation rate $\epsilon \propto X^2 \rho T^4$.

Problem 2.3. The condition for a resonant mode to exist is that an integral number of half wavelengths fits into the resonant cavity, for boundary conditions requiring a node at both ends of the cavity, that is $kL = n\pi$ where k is the wavevector and L is the length of the cavity. In the Sun the boundary conditions are more complicated, and the wave vector varies with the radius; therefore, the resonant condition in the JWKB approximation is:

$$\int_{\rm cavity} k_r dr = (n + \alpha)\pi$$

where k_r is the radial component of the wave vector and α is a constant (or a function of mode frequency) which depends on the boundary conditions. For acoustic p-modes k_r is determined from the dispersion relation (in which we neglect the acoustic cut-off frequency)

$$\omega^2 = k^2 c^2 = (k_r^2 + k_h^2) c^2$$

where ω is the mode frequency, k_h is the horizontal component of the wave vector, and c is the sound speed. The horizontal component k_h can be expressed in terms of the angular degree l of the corresponding spherical harmonics:

$$k_h = \sqrt{l(l+1)}/r \equiv L/r$$

where r is the radius, $L = \sqrt{l(l+1)}$.

(a) Derive the resonant condition for the p-modes:

$$\int_{r_t}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{dr}{c} = \frac{(n+\alpha)\pi}{\omega}, \tag{3}$$

and obtain equation for the radius of the lower turning point, r_t. What can you say about the location of the lower turning point for low-l and high-l modes?

(b) Consider the low-l limit ($l/n << 1$) and derive the formula for low-l mode frequencies:

$$\omega = \frac{(n + L/2 + \alpha)\pi}{\int_0^R \frac{dr}{c}} \tag{4}$$

assuming the sound speed is almost constant near the center. How does the oscillation spectrum look like if you observe the Sun as a star? Estimate the frequency separation between the modes assuming that the sound speed is constant inside the Sun.

(c) Estimate the frequency separation for α Ursae Majoris A observed with WIRE satellite, using data from Guenther et al. 2000, Astrophys. J. Letters, 530, L45. (extra credit)

(d) Consider the limit of high l assuming that the convection zone is fully adiabatic ($d\log P/dr = \gamma d\log\rho/dr$) and can be treated as a plane parallel layer, and the pressure, P, and density, ρ, are equal zero at the surface ($r = R$). Show that in this limit the oscillation frequencies are given by:

$$\omega^2 = \frac{2(\gamma - 1)g(n + \alpha)L}{R} \tag{5}$$

where γ is the adiabatic exponent, and g is the gravity acceleration at the solar surface. Make a plot of the $l - \nu$ diagram in the frequency range from 1 to 10 mHz, and the range of l from 100 to 2000. Include the f-mode ($\omega^2 = gk_h$) in the plot.
(R.F. Stein, On the modal structure of the solar oscillations, Astronomy and Astrophysics, vol. 105, no. 2, 1982, p. 417;
http://adsabs.harvard.edu/abs/1982A&A...105..417S)