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ORIGINAL ARTICLE

ADIPLS—the Aarhus adiabatic oscillation package

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Abstract Development of the Aarhus adiabatic pulsation code started around 1978. Although the main features have been stable for more than a decade, development of the code is continuing, concerning numerical properties and output. The code has been provided as a generally available package and has seen substantial use at a number of installations. Further development of the package, including bringing the documentation closer to being up to date, is planned as part of the HELAS Coordination Action.

Keywords Stars: oscillations · Numerical methods · Asteroseismology

1 Introduction

The goal of the development of the code was to have a simple and efficient tool for the computation of adiabatic oscillation frequencies and eigenfunctions for general stellar models, emphasizing also the accuracy of the results. Not surprisingly, given the long development period, the simplicity is now less evident. However, the code offers considerable flexibility in the choice of integration method as well as ability to determine all frequencies of a given model, in a given range of degree and frequency.

The choice of variables describing the equilibrium model and oscillations was to a large extent inspired by Dziembowski (1971). As discussed in Sect. 2.1 the equilibrium model is defined in terms of a minimal set of dimensionless variables, as well as by mass and radius of the model.

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Fairly extensive documentation of the code, on which the present paper in part is based, is provided with the distribution package.[1] Christensen-Dalsgaard and Berthomieu (1991) provided an extensive review of adiabatic stellar oscillations, emphasizing applications to helioseismology, and discussed many aspects and tests of the Aarhus package, whereas Christensen-Dalsgaard and Mullan (1994) carried out careful tests and comparisons of results on polytropic models; this includes extensive tables of frequencies which can be used for comparison with other codes.

2 Equations and numerical scheme

2.1 Equilibrium model

The equilibrium model is defined in terms of the following dimensionless variables:

$$\begin{aligned} x &\equiv r/R, \\ A_1 &\equiv q/x^3, \quad \text{where } q = m/M, \\ A_2 = V_g &\equiv -\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}\ln r} = \frac{Gm\rho}{\Gamma_1 pr}, \\ A_3 &\equiv \Gamma_1, \\ A_4 = A &\equiv \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}\ln r} - \frac{\mathrm{d}\ln \rho}{\mathrm{d}\ln r}, \\ A_5 = U &\equiv \frac{4\pi\rho r^3}{m}. \end{aligned} \tag{1}$$

[1] The package is available at
`http://astro.phys.au.dk/~jcd/adipack.n`

Here r is distance to the centre, m is the mass interior to r, R is the photospheric radius of the model and M is its mass; also, G is the gravitational constant, p is pressure, ρ is density, and $\Gamma_1 = (\partial \ln p/\partial \ln \rho)_{\rm ad}$, the derivative being at constant specific entropy. In addition, the model file defines M and R, as well as central pressure and density, in dimensional units, and scaled second derivatives of p and ρ at the centre (required from the expansions in the central boundary condition); finally, for models with vanishing surface pressure, assuming a polytropic relation between p and ρ in the near-surface region, the polytropic index is specified.

The following relations between the variables defined here and more "physical" variables are often useful:

$$p = \frac{GM^2}{4\pi R^4}\frac{x^2 A_1^2 A_5}{A_2 A_3}, \qquad \frac{{\rm d}p}{{\rm d}r} = -\frac{GM^2}{4\pi R^5} x A_1^2 A_5,$$
$$\rho = \frac{M}{4\pi R^3} A_1 A_5. \tag{2}$$

We may also express the characteristic frequencies for adiabatic oscillations in terms of these variables. Thus if N is the buoyancy frequency, S_l is the Lamb frequency at degree l and $\omega_{\rm a}$ is the acoustical cut-off frequency for an isothermal atmosphere, we have

$$N^2 = \frac{GM}{R^3}\hat{N}^2 = \frac{GM}{R^3} A_1 A_4, \tag{3}$$

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = \frac{GM}{R^3}\hat{S}_l^2 = \frac{GM}{R^3}\frac{l(l+1)A_1}{A_2}, \tag{4}$$

$$\omega_{\rm a}^2 = \frac{c^2}{4H_p^2} = \frac{GM}{R^3}\hat{\omega}_{\rm a}^2 = \frac{1}{4}\frac{GM}{R^3} A_1 A_2 A_3^2, \tag{5}$$

where c is the adiabatic sound speed, and $H_p = p/(g\rho)$ is the pressure scale height, g being the gravitational acceleration. Finally it may be noted that the squared sound speed is given by

$$c^2 = \frac{GM}{R}\hat{c}^2 = \frac{GM}{R} x^2 \frac{A_1}{A_2}. \tag{6}$$

These equations also define the dimensionless characteristic frequencies $\hat{N}$, $\hat{S}_l$ and $\hat{\omega}_{\rm a}$ as well as the dimensionless sound speed $\hat{c}$, which are often useful.

2.2 Formulation of the equations

As is well known the displacement vector of nonradial (spheroidal) modes can be written in terms of polar coordinates (r, θ, ϕ) as

$$\boldsymbol{\delta r} = {\rm Re}\left\{\left[\xi_r(r) Y_l^m(\theta,\phi)\mathbf{a}_r + \xi_{\rm h}(r)\left(\frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\phi\right)\right]\exp(-{\rm i}\omega t)\right\}. \tag{7}$$

Here $Y_l^m(\theta,\phi) = c_{lm} P_l^m(\cos\theta)\exp({\rm i}m\phi)$ is a spherical harmonic of degree l and azimuthal order m, θ being co-latitude and ϕ longitude; $P_l^m(x)$ is an associated Legendre function, and c_{lm} is a suitable normalization constant. Also, $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are unit vectors in the r, θ, and ϕ directions. Finally, t is time and ω is the angular frequency of the mode. Similarly, e.g., the Eulerian perturbation to pressure may be written[2]

$$p'(r,\theta,\phi,t) = {\rm Re}\left[p'(r) Y_l^m(\theta,\phi)\exp(-{\rm i}\omega t)\right]. \tag{8}$$

As the oscillations are adiabatic (and only conservative boundary conditions are considered) ω is real, and the amplitude functions $\xi_r(r)$, $\xi_{\rm h}(r)$, $p'(r)$, etc., can be chosen to be real.

The equations of adiabatic stellar oscillations, in the nonradial case, are expressed in terms of the following variables[3]:

$$y_1 = \frac{\xi_r}{R},$$
$$y_2 = x\left(\frac{p'}{\rho} + \Phi'\right)\frac{l(l+1)}{\omega^2 r^2} = \frac{l(l+1)}{R}\xi_{\rm h},$$
$$y_3 = -x\frac{\Phi'}{gr},$$
$$y_4 = x^2\frac{\rm d}{{\rm d}x}\left(\frac{y_3}{x}\right). \tag{9}$$

Here Φ' is the perturbation to the gravitational potential. Also, we introduce the dimensionless frequency σ by

$$\omega^2 = \frac{GM}{R^3}\sigma^2, \tag{10}$$

corresponding to (3–5). These quantities satisfy the following equations:

$$x\frac{{\rm d}y_1}{{\rm d}x} = (V_g - 2)y_1 + \left(1 - \frac{V_g}{\eta}\right)y_2 - V_g y_3, \tag{11}$$

$$x\frac{{\rm d}y_2}{{\rm d}x} = [l(l+1) - \eta A]y_1 + (A-1)y_2 + \eta A y_3, \tag{12}$$

$$x\frac{{\rm d}y_3}{{\rm d}x} = y_3 + y_4, \tag{13}$$

$$x\frac{{\rm d}y_4}{{\rm d}x} = -AU y_1 - U\frac{V_g}{\eta}y_2 + [l(l+1) + U(A-2) + UV_g]y_3 + 2(1-U)y_4. \tag{14}$$

[2] I do not here distinguish between the full perturbation and the radial amplitude function.

[3] The somewhat peculiar choice of y_3, y_4 results from the earlier use of an unconventional sign convention for Φ'; now, as usual, Φ' is defined such that the perturbed Poisson equation has the form $\nabla^2\Phi' = 4\pi G\rho'$, where ρ' is the Eulerian density perturbation.

Here $\eta = l(l+1)g/(\omega^2 r) = l(l+1)A_1/\sigma^2$, and the notation is otherwise as defined in (1). In the Cowling (1941) approximation, where the perturbation to the gravitational potential is neglected, the terms in y_3 are neglected in (11, 12) and (13, 14) are not used.

The dependent variables y_i in the nonradial case have been chosen in such a way that for $l > 0$ they all vary as x^{l-1} for $x \to 0$. For large l a considerable (and fundamentally unnecessary) computational effort would be needed to represent this variation sufficiently accurately with, e.g., a finite difference technique, if these variables were to be used in the numerical integration. Instead I introduce a new set of dependent variables by

$$\hat{y}_i = x^{-l+1} y_i, \quad i = 1, 2, 3, 4. \tag{15}$$

These variables are then $O(1)$ in x near the centre. They are used in the region where the variation in the y_i is dominated by the x^{l-1} behaviour, for $x < x_{\rm ev}$, say, where $x_{\rm ev}$ is determined on the basis of the asymptotic properties of the solution. This transformation permits calculating modes of arbitrarily high degree in a complete model.

For radial oscillations only y_1 and y_2 are used, where y_1 is defined as above, and

$$y_2 = \frac{p'}{\omega^2 R^2 \rho}. \tag{16}$$

Here the equations become

$$x\frac{{\rm d}y_1}{{\rm d}x} = (V_g - 2)y_1 - V_g\frac{\sigma^2 x^2}{q}y_2, \tag{17}$$

$$x\frac{{\rm d}y_2}{{\rm d}x} = \left[x - \frac{q}{\sigma^2 x^2}(A - U)\right]y_1 + A y_2. \tag{18}$$

The equations are solved on the interval $[x_1, x_{\rm s}]$ in x. Here, in the most common case involving a complete stellar model $x_1 = \epsilon$, where ϵ is a suitably small number such that the series expansion around $x = 0$ is sufficiently accurate; however, the code can also deal with envelope models with arbitrary x_1, typically imposing $\xi_r = 0$ at the bottom of the envelope. The outermost point is defined by $x_{\rm s} = R_{\rm s}/R$ where $R_{\rm s}$ is the surface radius, including the atmosphere; thus, typically, $x_{\rm s} > 1$.

2.3 Boundary conditions

The centre of the star, $r = 0$, is obviously a singular point of the equations. As discussed, e.g., by Christensen-Dalsgaard et al. (1974) boundary conditions at this point are obtained from a series expansion, in the present code to second significant order. In the general case this defines two conditions at the innermost non-zero point in the model. For radial oscillations, or nonradial oscillations in the Cowling approximation, one condition is obtained. The surface in a realistic model is typically defined at a suitable point in the stellar atmosphere, with non-zero pressure and density. Here the simple condition of vanishing Lagrangian pressure perturbation is implemented and sometimes used. However, more commonly a condition between pressure perturbation and displacement is established by matching continuously to the solution in an isothermal atmosphere extending continuously from the uppermost point in the model.[4] A very similar condition was presented by Unno et al. (1989). In addition, in the full nonradial case a condition is obtained from the continuous match of Φ' and its derivative to the vacuum solution outside the star.

In full polytropic models, or other models with vanishing surface pressure, the surface is also a singular point. In this case a boundary condition at the outermost non-singular point is obtained from a series expansion, assuming a near-surface polytropic behaviour (see Christensen-Dalsgaard and Mullan 1994, for details).

The code also has the option of considering truncated (e.g., envelope) models although at the moment only in the Cowling approximation or for radial oscillations. In this case the innermost boundary condition is typically the vanishing of the radial displacement ξ_r although other options are available.

2.4 Numerical scheme

The numerical problem can be formulated generally as that of solving

$$\frac{{\rm d}y_i}{{\rm d}x} = \sum_{j=1}^{I} a_{ij}(x) y_j(x), \quad \text{for } i = 1, \ldots, I, \tag{19}$$

with the boundary conditions

$$\sum_{j=1}^{I} b_{ij} y_j(x_1) = 0, \quad \text{for } i = 1, \ldots, I/2, \tag{20}$$

$$\sum_{j=1}^{I} c_{ij} y_j(x_{\rm s}) = 0, \quad \text{for } i = 1, \ldots, I/2. \tag{21}$$

Here the order I of the system is 4 for the full nonradial case, and 2 for radial oscillations or nonradial oscillations in the Cowling approximation. This system only allows nontrivial solutions for selected values of σ^2 which is thus an eigenvalue of the problem.

The programme permits solving these equations with two basically different techniques, each with some variants. The

[4] Note that since the frequency, and other variables, are taken to be real this can only be applied for frequencies below the acoustical cut-off frequency in the isothermal extension.

first is a shooting method, where solutions satisfying the boundary conditions are integrated separately from the inner and outer boundary, and the eigenvalue is found by matching these solutions at a suitable inner fitting point $x_{\rm f}$. The second technique is to solve the equations together with a normalization condition and all boundary conditions using a relaxation technique; the eigenvalue is then found by requiring continuity of one of the eigenfunctions at an interior matching point.

For simplicity I do not distinguish between $\hat{y}_i$ and y_i (cf. Sect. 2.2) in this section. It is implicitly understood that the dependent variable (which is denoted y_i) is $\hat{y}_i$ for $x < x_{\rm ev}$ and y_i for $x \geq x_{\rm ev}$. The numerical treatment of the transition between $\hat{y}_i$ and y_i has required a little care in the coding.

2.5 The shooting method

It is convenient here to distinguish between $I = 2$ and $I = 4$. For $I = 2$ the differential equations (19) have a unique (apart from normalization) solution $y_i^{(\rm i)}$ satisfying the inner boundary conditions (20), and a unique solution $y_i^{(\rm o)}$ satisfying the outer boundary conditions (21). These are obtained by numerical integration of the equations. The final solution can then be represented as $y_j = C^{(\rm i)} y_j^{(\rm i)} = C^{(\rm o)} y_j^{(\rm o)}$. The eigenvalue is obtained by requiring that the solutions agree at a suitable matching point $x = x_{\rm f}$, say. Thus

$$\begin{aligned} C^{(\rm i)} y_1^{(\rm i)}(x_{\rm f}) &= C^{(\rm o)} y_1^{(\rm o)}(x_{\rm f}), \\ C^{(\rm i)} y_2^{(\rm i)}(x_{\rm f}) &= C^{(\rm o)} y_2^{(\rm o)}(x_{\rm f}). \end{aligned} \tag{22}$$

These equations clearly have a non-trivial solution $(C^{(\rm i)}, C^{(\rm o)})$ only when their determinant vanishes, i.e., when

$$\Delta = y_1^{(\rm i)}(x_{\rm f}) y_2^{(\rm o)}(x_{\rm f}) - y_2^{(\rm i)}(x_{\rm f}) y_1^{(\rm o)}(x_{\rm f}) = 0. \tag{23}$$

Equation (23) is therefore the eigenvalue equation.

For $I = 4$ there are two linearly independent solutions satisfying the inner boundary conditions, and two linearly independent solutions satisfying the outer boundary conditions. The former set $\{y_i^{(\rm i,1)}, y_i^{(\rm i,2)}\}$ is chosen by setting

$$\begin{aligned} y_1^{(\rm i,1)}(x_1) &= 1, \qquad y_3^{(\rm i,1)}(x_1) = 0, \\ y_1^{(\rm i,2)}(x_1) &= 1, \qquad y_3^{(\rm i,2)}(x_1) = 1, \end{aligned} \tag{24}$$

and the latter set $\{y_i^{(\rm o,1)}, y_i^{(\rm o,2)}\}$ is chosen by setting

$$\begin{aligned} y_1^{(\rm o,1)}(x_{\rm s}) &= 1, \qquad y_3^{(\rm o,1)}(x_{\rm s}) = 0, \\ y_1^{(\rm o,2)}(x_{\rm s}) &= 1, \qquad y_3^{(\rm o,2)}(x_{\rm s}) = 1. \end{aligned} \tag{25}$$

The inner and outer boundary conditions are such that, given y_1 and y_3, y_2 and y_4 may be calculated from them; thus (25) and (26) completely specify the solutions, which are obtained by integrating from the inner or outer boundary. The final solution can then be represented as

$$y_j = C^{(\rm i,1)} y_j^{(\rm i,1)} + C^{(\rm i,2)} y_j^{(\rm i,2)} = C^{(\rm o,1)} y_j^{(\rm o,1)} + C^{(\rm o,2)} y_j^{(\rm o,2)}. \tag{26}$$

At the fitting point $x_{\rm f}$ continuity of the solution requires that

$$\begin{aligned} &C^{(\rm i,1)} y_j^{(\rm i,1)}(x_{\rm f}) + C^{(\rm i,2)} y_j^{(\rm i,2)}(x_{\rm f}) \\ &\quad = C^{(\rm o,1)} y_j^{(\rm o,1)}(x_{\rm f}) + C^{(\rm o,2)} y_j^{(\rm o,2)}(x_{\rm f}), \quad j = 1, 2, 3, 4. \end{aligned} \tag{27}$$

This set of equations only has a non-trivial solution if

$$\Delta = \det \begin{Bmatrix} y_{1,\rm f}^{(\rm i,1)} & y_{1,\rm f}^{(\rm i,2)} & y_{1,\rm f}^{(\rm o,1)} & y_{1,\rm f}^{(\rm o,2)} \\ y_{2,\rm f}^{(\rm i,1)} & y_{2,\rm f}^{(\rm i,2)} & y_{2,\rm f}^{(\rm o,1)} & y_{2,\rm f}^{(\rm o,2)} \\ y_{3,\rm f}^{(\rm i,1)} & y_{3,\rm f}^{(\rm i,2)} & y_{3,\rm f}^{(\rm o,1)} & y_{3,\rm f}^{(\rm o,2)} \\ y_{4,\rm f}^{(\rm i,1)} & y_{4,\rm f}^{(\rm i,2)} & y_{4,\rm f}^{(\rm o,1)} & y_{4,\rm f}^{(\rm o,2)} \end{Bmatrix} = 0, \tag{28}$$

where, e.g., $y_{j,\rm f}^{(\rm i,1)} \equiv y_j^{(\rm i,1)}(x_{\rm f})$. Thus (28) is the eigenvalue equation in this case.

Clearly Δ as defined in either (23) or (28) is a smooth function of σ^2, and the eigenfrequencies are found as the zeros of this function. This is done in the programme using a standard secant technique. However, the programme also has the option for scanning through a given interval in σ^2 to look for changes of sign of Δ, possibly iterating for the eigenfrequency at each change of sign. Thus it is possible to search a given region of the spectrum completely automatically.

The programme allows the use of two different techniques for solving the differential equations. One is the standard second-order centred difference technique, where the differential equations are replaced by the difference equations

$$\frac{y_i^{n+1} - y_i^n}{x^{n+1} - x^n} = \frac{1}{2} \sum_{j=1}^{I} \left[a_{ij}^n y_j^n + a_{ij}^{n+1} y_j^{n+1} \right], \quad i = 1, \ldots, I. \tag{29}$$

Here I have introduced a mesh $x_1 = x^1 < x^2 < \cdots < x^N = x_{\rm s}$ in x, where N is the total number of mesh points; $y_i^n \equiv y_i(x^n)$, and $a_{ij}^n \equiv a_{ij}(x^n)$. These equations allow the solution at $x = x^{n+1}$ to be determined from the solution at $x = x^n$.

The second technique was proposed by Gabriel and Noels (1976); here on each mesh interval (x^n, x^{n+1}) we consider the equations

$$\frac{{\rm d}y_i^{(n)}}{{\rm d}x} = \sum_{j=1}^{I} \bar{a}_{ij}^n y_j^{(n)}(x), \quad \text{for } i = 1, \ldots, I, \tag{30}$$

with constant coefficients, where $\bar{a}_{ij}^n = 1/2(a_{ij}^n + a_{ij}^{n+1})$. These equations may be solved analytically on the mesh intervals, and the complete solution is obtained by continuous matching at the mesh points. This technique clearly permits the computation of solutions varying arbitrarily rapidly, i.e., the calculation of modes of arbitrarily high order. On the other hand solving (30) involves finding the eigenvalues and eigenvectors of the coefficient matrix, and therefore becomes very complex and time consuming for higher-order systems. Thus in practice it has only been implemented for systems of order 2, i.e., radial oscillations or nonradial oscillations in the Cowling approximation.

2.6 The relaxation technique

If one of the boundary conditions is dropped, the difference equations, with the remaining boundary condition and a normalization condition, constitute a set of linear equations for the $\{y_j^n\}$ which can be solved for any value of σ; this set may be solved efficiently by forward elimination and backsubstitution (e.g. Baker et al. 1971), with a procedure very similar to the so-called Henyey technique (e.g., Henyey et al. 1959; see also Christensen-Dalsgaard 2007) used in stellar modelling. The eigenvalue is then found by requiring that the remaining boundary condition, which effectively takes the role of $\Delta(\sigma)$, be satisfied. However, as both boundaries, at least in a complete model, are either singular or very nearly singular, the removal of one of the boundary conditions tends to produce solutions that are somewhat ill-behaved, in particular for modes of high degree. This in turn is reflected in the behaviour of Δ as a function of σ.

This problem is avoided in a variant of the relaxation technique where the difference equations are solved separately for $x \leq x_\mathrm{f}$ and $x \geq x_\mathrm{f}$, by introducing a double point $x_\mathrm{f}^- = x^{n_\mathrm{f}} = x^{n_\mathrm{f}+1} = x_\mathrm{f}^+$ in the mesh. The solution is furthermore required to satisfy the boundary conditions (20) and (21), a suitable normalization condition (e.g. $y_1(x_\mathrm{s}) = 1$), and continuity of all but one of the variables at $x = x_\mathrm{f}$, e.g.,

$$\begin{aligned} y_1(x_\mathrm{f}^-) &= y_1(x_\mathrm{f}^+), \\ y_3(x_\mathrm{f}^-) &= y_3(x_\mathrm{f}^+), \\ y_4(x_\mathrm{f}^-) &= y_4(x_\mathrm{f}^+), \end{aligned} \tag{31}$$

(when $I = 2$ clearly only the first continuity condition is used). We then set

$$\Delta = y_2(x_\mathrm{f}^-) - y_2(x_\mathrm{f}^+), \tag{32}$$

and the eigenvalues are found as the zeros of Δ, regarded as a function of σ^2. With this definition, Δ may have singularities with discontinuous sign changes that are not associated with an eigenvalue, and hence a little care is required in the search for eigenvalues. However, close to an eigenvalue Δ is generally well-behaved, and the secant iteration may be used without problems.

As implemented here the shooting technique is considerably faster than the relaxation technique, and so should be used whenever possible (notice that both techniques may use the difference equations (29) and so they are numerically equivalent, in regions of the spectrum where they both work). For *second-order systems* the shooting technique can probably always be used; the integrations of the inner and outer solutions should cause no problems, and the matching determinant in (23) is well-behaved. For *fourth-order systems*, however, this needs not be the case. For modes where the perturbation to the gravitational potential has little effect on the solution, the two solutions $y_j^{(\mathrm{i},1)}$ and $y_j^{(\mathrm{i},2)}$, and similarly the two solutions $y_j^{(\mathrm{o},1)}$ and $y_j^{(\mathrm{o},2)}$, are almost linearly dependent, and so the matching determinant nearly vanishes for any value of σ^2. This is therefore the situation where the relaxation technique may be used with advantage. This applies, in particular, to the calculation of modes of moderate and high degree which are essential to helioseismology.

2.7 Improving the frequency precision

To make full use of the increasingly accurate observed frequencies the computed frequencies should clearly at the very least match the observational accuracy, for a given model. Only in this way do the frequencies provide a faithful representation of the properties of the model, in comparisons with the observations. However, since the numerical errors in the computed frequencies are typically highly systematic, they may affect the asteroseismic inferences even if they are smaller than the random errors in the observations, and hence more stringent requirements should be imposed on the computations. Also, the fact that solar-like oscillations, and several other types of asteroseismically interesting modes, tend to be of high radial order complicates reaching the required precision.

The numerical techniques discussed so far are generally of second order. This yields insufficient precision in the evaluation of the eigenfrequencies, unless a very dense mesh is used in the computation (see also Moya et al. 2007). The code may apply two techniques to improve the precision.

One technique (cf. Christensen-Dalsgaard 1982) uses the fact that the frequency approximately satisfies a variational principle (Chandrasekhar 1964).[5] The variational expression may formally be written as

$$\sigma^2 = \sigma_\mathrm{var}^2 \equiv \Sigma(\xi)^2 = \frac{\mathcal{K}(\xi)}{\mathcal{I}(\xi)}, \tag{33}$$

[5]The variational principle is exact, formally, when the surface Lagrangian pressure perturbation is set to zero, but not when the match to an isothermal atmosphere is used.

where $\mathcal{K}$ and $\mathcal{I}$ are integrals over the equilibrium model depending on the eigenfunction, here represented by ξ. The variational property implies that any error $\delta\xi$ in ξ induces an error in Σ^2 that is $\mathcal{O}(|\delta\xi|^2)$. Thus by substituting the computed eigenfunction into the variational expression a more precise determination of σ^2 should result. This has indeed been confirmed (Christensen-Dalsgaard 1982; Christensen-Dalsgaard and Berthomieu 1991; Christensen-Dalsgaard and Mullan 1994).

The second technique uses explicitly that the difference scheme (29), which is used by one version of the shooting technique, and the relaxation technique, is of second order. Consequently the truncation errors in the eigenfrequency and eigenfunction scale as N^{-2}. If $\sigma(N/2)$ and $\sigma(N)$ are the eigenfrequencies obtained from solutions with $N/2$ and N meshpoints, the leading-order error term therefore cancels in

$$\sigma_{\mathrm{Ri}} = \frac{1}{3}\left[4\sigma(N) - \sigma\left(\frac{1}{2}N\right)\right]. \tag{34}$$

This procedure, known as *Richardson extrapolation*, was used by Shibahashi and Osaki (1981). It provides an estimate of the eigenfrequency that is substantially more accurate than $\sigma(N)$, although of course at some added computational expense. Indeed, since the error in the representation (29) depends only on even powers of N^{-1}, the leading term of the error in σ_{Ri} is $\mathcal{O}(N^{-4})$.

Even with these techniques the precision of the computed frequencies may be inadequate if the mesh used in stellar-evolution calculations is used also for the computation of the oscillations. The number of meshpoints is typically relatively modest and the distribution may not reflect the requirement to resolve properly the eigenfunctions of the modes. Christensen-Dalsgaard and Berthomieu (1991) discussed techniques to redistribute the mesh in a way that takes into account the asymptotic behaviour of the eigenfunctions; a code to do so, based on four-point Lagrangian interpolation, is included in the ADIPLS distribution package. On the other hand, for computing low-order modes (as are typically relevant for, say, δ Scuti or β Cephei stars), the original mesh of the evolution calculation may be adequate.

It is difficult to provide general recommendations concerning the required number of points or the need for redistribution, since this depends strongly on the types of modes and the properties of the stellar model. It is recommended to carry out experiments varying the number and distribution of points to obtain estimates of the intrinsic precision of the computation (e.g., Christensen-Dalsgaard and Berthomieu 1991; Christensen-Dalsgaard and Mullan 1994). In the latter case, considering simple polytropic models, it was found that 4801 points yielded a relative precision substantially better than 10^{-6} for high-order p-modes, when Richardson extrapolation was used.

In the discussion of the frequency calculation it is important to distinguish between *precision* and *accuracy*, the latter obviously referring to the extent to which the computed frequencies represent what might be considered the 'true' frequencies of the model. In particular, the manipulations required to derive (33) and to demonstrate its variational property depend on the equation of hydrostatic support being satisfied. If this is not the case, as might well happen in an insufficiently careful stellar model calculation, the value determined from the variational principle may be quite precise, in the sense of numerically stable, but still unacceptably far from the correct value. Indeed, a comparison between σ_{var} and σ_{Ri} provides some measure of the reliability of the computed frequencies (e.g. Christensen-Dalsgaard and Berthomieu 1991).

3 Computed quantities

The programme finds the order of the mode according to the definition developed by Scuflaire (1974) and Osaki (1975), based on earlier work by Eckart (1960). Specifically, the order is defined by

$$n = -\sum_{x_{z1}>0} \mathrm{sign}\left(y_2 \frac{\mathrm{d}y_1}{\mathrm{d}x}\right) + n_0. \tag{35}$$

Here the sum is over the zeros $\{x_{z1}\}$ in y_1 (excluding the centre), and sign is the sign function, $\mathrm{sign}(z) = 1$ if $z > 0$ and $\mathrm{sign}(z) = -1$ if $z < 0$. For a complete model that includes the centre $n_0 = 1$ for radial oscillations and $n_0 = 0$ for nonradial oscillations. Thus the lowest-order radial oscillation has order $n = 1$. Although this is contrary to the commonly used convention of assigning order 0 to the fundamental radial oscillation, the convention used here is in fact the more reasonable, in the sense that it ensures that n is invariant under a continuous variation of l from 0 to 1. With this definition $n > 0$ for p modes, $n = 0$ for f modes, and $n < 0$ for g modes, at least in simple models.

It has been found that this procedure has serious problems for dipolar modes in centrally condensed models (e.g., Lee 1985; Guenther 1991; Christensen-Dalsgaard and Mullan 1994). The eigenfunctions y_1 are shifted such that nodes disappear or otherwise provide spurious results when (35) is used to determine the mode order. A procedure that does not suffer from this difficulty has recently been developed by Takata (2006b); I discuss it further in Sect. 4.

A powerful measure of the characteristics of a mode is provided by the *normalized inertia*. The code calculates this as

$$\hat{E} = \frac{\int_{r_1}^{R_s} [\xi_r^2 + l(l+1)\xi_{\mathrm{h}}^2]\rho r^2 \mathrm{d}r}{M[\xi_r(R_{\mathrm{phot}})^2 + l(l+1)\xi_{\mathrm{h}}(R_{\mathrm{phot}})^2]}$$

$$= \frac{\int_{x_1}^{x_s} \left[y_1^2 + y_2^2/l(l+1)\right] qU \,\mathrm{d}x/x}{4\pi[y_1(x_{\rm phot})^2 + y_2(x_{\rm phot})^2/l(l+1)]}. \tag{36}$$

(For radial modes the terms in y_2 are not included.) Here $r_1 = Rx_1$ and $R_s = Rx_s$ are the distance of the innermost mesh point from the centre and the surface radius, respectively, and $x_{\rm phot} = R_{\rm phot}/R = 1$ is the fractional photospheric radius. The normalization at the photosphere is to some extent arbitrary, of course, but reflects the fact that many radial-velocity observations use lines formed relatively deep in the atmosphere. A more common definition of the inertia is

$$E = 4\pi\hat{E} = \frac{M_{\rm mode}}{M}, \tag{37}$$

where $M_{\rm mode}$ is the so-called *mode mass*.

The code has the option to output the eigenfunctions, in the form of $\{y_j(x^n)\}$. In addition (or instead) the displacement eigenfunctions can be output in a form indicating the region where the mode predominantly resides, in an energetical sense, as

$$\begin{aligned} z_1(x) &= \left(\frac{4\pi r^3\rho}{M}\right)^{1/2} y_1(x) = \left(\frac{4\pi r^3\rho}{M}\right)^{1/2} \frac{\xi_r(r)}{R}, \\ z_2(x) &= \frac{1}{\sqrt{l(l+1)}} \left(\frac{4\pi r^3\rho}{M}\right)^{1/2} y_2(x) \\ &= \sqrt{l(l+1)} \left(\frac{4\pi r^3\rho}{M}\right)^{1/2} \frac{\xi_{\rm h}(r)}{R} \end{aligned} \tag{38}$$

(for radial modes only z_1 is found). These are defined in such a way that

$$\hat{E} = \frac{\int_{x_1}^{x_s} [z_1^2 + z_2^2] \mathrm{d}x/x}{4\pi[y_1(x_{\rm phot})^2 + y_2(x_{\rm phot})^2/l(l+1)]}. \tag{39}$$

The form provided by the z_i is also convenient, e.g., for computing rotational splittings $\delta\omega_{nlm} = \omega_{nlm} - \omega_{nl0}$ (e.g., Gough 1981), where ω_{nlm} is the frequency of a mode of radial order n, degree l and azimuthal order m. For slow rotation the splittings are obtained from first-order perturbation analysis as

$$\delta\omega_{nlm} = m \int_0^{R_s} \int_0^{\pi} K_{nlm}(r,\theta) \Omega(r,\theta) r \,\mathrm{d}r \,\mathrm{d}\theta, \tag{40}$$

characterized by *kernels* K_{nlm}, where in general the angular velocity Ω depends on both r and θ. The code has built in the option to compute kernels for first-order rotational splitting in the special case where Ω depends only on r.

4 Further developments

Several revisions of the code have been implemented in preliminary form or are under development. A substantial improvement in the numerical solution of the oscillation equations, particularly for high-order modes, is the installation of a fourth-order integration scheme, based on the algorithm of Cash and Moore (1980). This is essentially operational but has so far not been carefully tested. Comparisons with the results of the variational expression and the use of Richardson extrapolation, of the same formal order, will be particularly interesting.

As discussed by Moya et al. (2007) the use of p' (or, as here, $\xi_{\rm h}$) as one of the integration variables has the disadvantage that the quantity A enters into the oscillation equations. In models with a density discontinuity, such as results if the model has a growing convective core and diffusion is neglected, A has a delta-function singularity at the point of the discontinuity. In the ADIPLS calculations this is dealt with by replacing the discontinuity by a very steep and well-resolved slope. However, it would obviously be an advantage to avoid this problem altogether. This can be achieved by using instead the Lagrangian pressure perturbation δp as one of the variables. Implementing this option would be a relatively straightforward modification to the code and is under consideration.

The proper classification of dipolar modes of low order in centrally condensed models has been a long-standing problem in the theory of stellar pulsations, as discussed in Sect. 3. Such a scheme must provide a unique order for each mode, such that the order is invariant under continuous changes of the equilibrium model, e.g., as a result of stellar evolution. As a major breakthrough, Takata in a series of papers has elucidated important properties of these modes and defined a new classification scheme satisfying this requirement (Takata 2005, 2006a, 2006b). A preliminary version of this scheme has been implemented and tested; however, the latest and most convenient form of the Takata classification still needs to be installed.

A version of the code has been established which computes the first-order rotational splitting for a given rotation profile $\Omega(r)$, in addition to setting up the corresponding kernels. This is being extended by K. Burke, Sheffield, to cover also second-order effects of rotation, based on the formalism of Gough and Thompson (1990). An important motivation for this is the integration, discussed by Christensen-Dalsgaard (2007), of the pulsation calculation with the ASTEC evolution code to allow full calculation of oscillation frequencies for a model of specified parameters (mass, age, initial rotation rate, etc.) as the result of a single subroutine call.

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