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Detectability of Large-Scale Solar Subsurface Flows

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Abstract
The accuracy of helioseismic measurement is limited by the stochastic nature of solar oscillations. In this article I use a Gaussian statistical model of the global seismic wave field of the Sun to investigate the noise limitations of direct-modeling analysis of convection-zone-scale flows. The theoretical analysis of noise is based on hypothetical data which cover the entire photosphere, including the portions invisible from the Earth. Noise estimates are derived for measurements of the flow-dependent couplings of global-oscillation modes and for combinations of coupling measurements which isolate vector-spherical-harmonic components of the flow velocity. For current helioseismic observations, which sample only a fraction of the photosphere, the inferred detection limits are best regarded as upper limits. The flow-velocity fields considered in this work are assumed to be decomposable into vector-spherical-harmonic functions of degree less than 5. The problem of measuring the general velocity field is shown to be similar enough to the well-studied problem of measuring differential rotation to permit rough estimates of flow detection thresholds to be gleaned from past helioseismic analysis. I estimate that, with existing and anticipated helioseismic datasets, large-scale flow-velocity amplitudes of a few tens of $\mathrm{m\,s^{-1}}$ should be detectable near the base of the convection zone.

Keywords: Solar interior; mass flows; helioseismology

1. Introduction

The largest-scale mass flows in the Sun include the differential rotation (or "angular velocity"), meridional flow, and *giant-cell* convection. Gradients of the subsurface angular velocity, particularly in the tachocline layer near the base of the convection zone, are thought to drive the magnetic dynamo. Axisymmetric meridional flow, which close to the photosphere is observed as a motion toward the heliographic poles, plays an important role in flux transport dynamos (Sheeley, 2005; Dikpati and Gilman, 2006). The largest scales of solar convection are believed to participate in transporting the Sun's luminosity through the convection zone and in maintaining the differential rotation. The

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most stringent observational constraints on the angular velocity are provided by whole-Sun oscillation-frequency-splitting measurements (Thompson *et al.*, 2003; Howe, 2009). For other large-scale flows, the methods of local helioseismology are used (Gizon and Birch, 2005).

Numerical simulations of convection-zone-scale flows are needed for a detailed understanding of the dynamic and magnetic solar interior. While numerical simulations have been able to reproduce the main features of the angular velocity, they have had less success in describing meridional flow and large scales of turbulence (Miesch, 2005). The meridional flow-velocity seen in simulations varies more rapidly both in space and in time than the velocity pattern that has emerged from observational analysis. Large-scale turbulent motions are seen in the first ~ 15 Mm below the photosphere (Hathaway *et al.*, 2000; Featherstone *et al.*, 2006), though the observed motions show no distinct giant-cell scale, unlike, say, the supergranulation scale. Turbulent motions have yet to be seen at greater depths, but observational upper limits have been placed on the flow speeds which are at least an order of magnitude smaller than the speeds seen in simulations. The limits therefore pose a major challenge for theory (Hanasoge, Duvall, and Sreenivasan, 2012; Gizon and Birch, 2012).

A rigorous noise model is crucial to the helioseismic detection of weak flows. This article presents a theoretical derivation of the noise expected in seismic measurements of the flow velocity on large spatial scales. The noise calculation is predicated on direct-modeling analysis (Woodard, 2007, 2009) of ideal observations which isolate the signals of individual *p*- and *f*-mode oscillations of the entire Sun. For conceptual clarity, the data analysis considered in the noise derivation proceeds in stages, starting with previously-developed expressions for the statistical expectations and covariances of the signal "covariance" data for a Sun with weak, slowly-varying mass flows. In the first stage of the data analysis, the coefficients which quantify the flow-dependent pairwise coupling of oscillation modes are obtained from a simple least-squares fit to covariance data. Orthonormal combinations of the estimated coupling coefficients are then computed, such that each projected coupling is sensitive to a single spherical-harmonic- and temporal-frequency component of either the poloidal or the toroidal part of the flow velocity. In the final stage of the data-analysis procedure, the projected mode-coupling data are inverted for the subsurface flow velocity. The present article describes approximations which considerably simplify the form of the helioseismic forward and noise models. In particular, an approximation is made for the form of the sensitivity of the projected coupling data to the flow velocity, appropriate for modes of large spherical-harmonic degree. In this approximation, the problems of inverting individual spherical-harmonic components of the flow-velocity degenerate into the well-studied problem of inverting mode-frequency-splitting coefficients for the angular velocity. The approximation therefore permits rough estimates of general large-scale flow-velocity measurement noise to be inferred from studies of the angular velocity inversion problem.

2. Basic Assumptions

As a basis for evaluating statistical uncertainty in the measurement of large-scale solar flows, I use "statistical-waveform" direct-modeling analysis of solar oscillation data, as described in Woodard (2009, hereafter W09). Helioseismic statistical-waveform analyses using covariance data in the form $\varphi'\varphi^*$, where φ and φ' are frequency-wavevector components of the photospheric Doppler velocity field, have been carried out for both large portions of the solar disk, taking spherical geometry into account, and for small patches, assuming Cartesian geometry. In contrast to the analyses which have been performed on solar data, the version of waveform analysis considered here also uses purely power-spectral ($\varphi' = \varphi$) data.

For general helioseismic analysis of large regions of the disk, it is common to work with the coefficients $\varphi \equiv \varphi_\omega^{\ell m}$ of a spherical-harmonic (ℓ, m) and temporal-frequency (ω) decomposition of the photospheric oscillation signal. The decomposition of the surface wave field into spherical-harmonic components nominally segregates the signal due to modes of different degree ℓ and azimuthal order m, while the temporal-frequency decomposition further isolates the modes according to radial order n. Segregation of mode signals greatly facilitates the task of characterizing important properties, like frequency and linewidth, of global oscillation modes. Helioseismic waveform analysis, and the closely-related analysis of global-mode eigenfunctions Woodard *et al.* (2012, hereafter W12), complement the traditional mode-frequency analysis by providing information about the flow-dependent couplings of distinct oscillation modes.

Existing solar oscillation datasets sample only the Earth-facing portion of the photosphere and the resulting aliasing of mode signals complicates helioseismic analysis. This article presents a theoretical analysis of the noise of helioseismic flow measurements by considering hypothetical observations which uniformly sample the entire photosphere. While the primary motive for focusing on ideal data is to reduce the complexity of the analysis without sacrificing too much realism, additional motivation comes from the possibility that better coverage of the photosphere will eventually be obtainable. As a further simplification, it is assumed that, for given (ℓ, m), the $\varphi_\omega^{\ell m}$ used in waveform analysis are taken from a narrow frequency interval close to a mode frequency $[\omega_{n\ell m}]$ for some n. This frequency selectivity, which serves to segregate the signals from modes of different n, is not expected to significantly compromise the overall signal-to-noise of the flow measurements, as the "inter-ridge" data which are ignored turn out to be fairly insensitive to the flow velocity. Therefore the $\varphi_\omega^{\ell m}$ can be used as proxies for the amplitudes $a_\omega^{n\ell m}$ of individual modes, defined as the coefficients in an expansion of the wave field in global-mode eigenfunctions (e.g. W12). In what follows, φ_ω^α will denote the signal which is mainly sensitive to the mode $\alpha = (n, \ell, m)$. It will be assumed that φ_ω^α has been scaled to the mode amplitude a_ω^α plus a background signal.

Theoretical expressions for the data expectations and covariances used in linearized waveform inversions are given in W09. The expressions are conveniently separated into zeroth-order, flow-independent, parts and flow-dependent perturbations. In zeroth order, and in the approximation where modes are excited independently, the only non-zero data expectations are the power spectra $P^\alpha_\omega \equiv E[|\varphi^\alpha_\omega|^2]_0$, which will be taken to have the form

$$P^\alpha_\omega = E[|a^\alpha_\omega|^2]_0 + B^\alpha_\omega. \tag{1}$$

The first term is the spectrum of the mode amplitude, while the second term represents the spectrum of background signals. The mode-amplitude spectrum is expected to have an approximately Lorentzian profile, as the zeroth-order mode amplitude is

$$a^\alpha_{\omega,0} = R^\alpha_\omega \, x^\alpha_\omega, \tag{2}$$

where

$$R^\alpha_\omega \approx -[2\omega_\alpha(\omega - \omega_\alpha + \mathrm{i}\frac{\gamma_\alpha}{2})]^{-1} \tag{3}$$

is the frequency-domain response of a simple oscillator of frequency ω_α and damping rate γ_α to a source function x^α_ω. The source function is assumed to be "broad-band", in the sense of being a weak function of ℓ, m, and ω.

Flows and other aspherical perturbations induce correlations between different spectral-domain components of the seismic signal. The first-order dependence of the data expectations on the flow velocity is approximately

$$\delta E[\varphi^{\alpha'}_{\omega'}\varphi^{\alpha\,*}_{\omega}] = -2\omega\lambda^{\alpha'}_{\alpha}(\omega' - \omega)\, R^{\alpha'}_{\omega'}\, E[|a^\alpha_\omega|^2]_0 + \dagger, \tag{4}$$

where the second term, indicated by "†", is obtained by replacing the indices $\alpha\omega$ and $\alpha'\omega'$ in and complex conjugating the first term. The complex-valued coefficient

$$\lambda^{\alpha'}_{\alpha}(\omega) = -\mathrm{i}\int_\odot \mathrm{d}m\, \boldsymbol{\xi}^*_{\alpha'} \cdot (\boldsymbol{u}_\omega \cdot \nabla \boldsymbol{\xi}_\alpha), \tag{5}$$

in which dm denotes an element of mass, is a measure of the strength with which the modes α and α' are dynamically coupled by the frequency component $\boldsymbol{u}_\omega$ of the flow velocity field. This work uses the same normalization convention for the mode eigenfunctions $[\boldsymbol{\xi}_\alpha]$ and the same discrete Fourier frequency convention as W09. Since solar flows are ignored in zeroth order, the eigenfunctions to be used in evaluating Equation (5) are those of a static, spherically-symmetric star. The above expression for the signal covariance perturbation ignores the influence of the flow velocity on the background contribution.

An integration-by-parts of the right side of Equation 5 using the mass-flux constraint $\nabla \cdot (\rho \mathbf{u}) = 0$, which is expected to be a good approximation for large-scale flows, and the condition $\mathbf{u}_{-\omega} = \mathbf{u}^*_\omega$ for a real-valued flow velocity yields the Hermitian property

$$\lambda^{\alpha *}_{\alpha'}(-\omega) = \lambda^{\alpha'}_{\alpha}(\omega). \tag{6}$$

With this approximation, Equation (4) becomes

$$\delta E[\varphi^{\alpha'}_{\omega'}\varphi^{\alpha\,*}_{\omega}] \approx (-2\omega\, R^{\alpha'}_{\omega'}\, E[|a^\alpha_\omega|^2]_0 + \dagger)\, \lambda^{\alpha'}_{\alpha}(\omega' - \omega). \tag{7}$$

The present analysis of error builds on earlier treatments (*e.g.* Gizon and Birch, 2004; Woodard, 2007) where a Gaussian distribution was assumed for the observed signal and the data covariances are computed in zeroth order. With these approximations, the covariance matrix of the data, regarded as a set of real and imaginary parts, is purely diagonal. For the variances of $y \equiv \varphi' \varphi *$, one obtains

$$\sigma^2[\mathrm{Re}(y)] = \sigma^2[\mathrm{Im}(y)] = \frac{PP'}{2}, \tag{8}$$

for distinct φ, φ', supplementing the more-familiar relation

$$\sigma^2(y) = P^2, \tag{9}$$

for identical φ, φ', where $P \equiv E[|\varphi|^2]_0$ and $P' \equiv E[|\varphi'|^2]_0$.

3. Coupling-Coefficient Estimates and Their Uncertainties

It is illuminating to conceptually divide the task of estimating subsurface flow into two subtasks: 1) estimate the coupling coefficients $[\lambda_\alpha^{\alpha'}(\omega)]$ and their covariances from the signal covariance data and their covariance matrix and 2) invert the derived coupling coefficients for the subsurface flow velocity. This approach generalizes conventional mode-frequency analysis, which deals only with the self-couplings $[\lambda_\alpha^\alpha(\omega = 0)]$.

The main goal of this section is to derive approximate expressions for coupling-coefficient estimates and their covariances. Though it may not be optimal from the standpoint of signal to noise, it is useful to adopt a least-squares fitting procedure, as a basis for estimating coupling coefficients. For simplicity, I consider a hypothetical fitting in which only the coupling parameters are varied, other parameters (e.g. mode linewidth and power) on which the covariance data depend being frozen at their true values. Error analysis based on such a consideration will tend to overestimate the accuracy of real coupling-constant measurements, as the problems of determining all the relevant parameters are interlinked. The diagonal structure of the data covariance matrix and the fact that an individual covariance datum is sensitive to precisely one coupling parameter implies that the coupling parameters can be estimated independently of one another and that their covariance matrix is diagonal.

Equation (7), describing data sensitivity, can be streamlined for the problem of estimating the coupling parameter $\lambda \equiv \lambda_\alpha^{\alpha'}(\sigma)$, using the shorthand φ,φ',R,R',... to denote φ_ω^α,$\varphi_{\omega'}^{\alpha'}$, R_ω^α,$R_{\omega'}^{\alpha'}$,..., where $\omega' = \omega + \sigma$, and by defining residual covariance data $y_\omega \equiv \varphi' \varphi^* - E[\varphi' \varphi^*]_0$. Thus $\delta E[\varphi' \varphi^*] = E[y_\omega]$. The form of the data covariance matrix, described by Equations (8) and (9), further simplifies through the use of scaled residual data $z_\omega \equiv y_\omega / \sqrt{PP'}$. With these definitions and notational changes, the sensitivity to the coupling parameter becomes

$$E[z_\omega] = K_\omega \, \lambda, \tag{10}$$

where

$$K_\omega = -2(\omega R' E[|a|^2]_0 + \omega' R^* E[|a'|^2]_0)) / \sqrt{P'P}. \tag{11}$$

Therefore the least-squares estimate of λ is

$$\hat{\lambda} = \sum_{\omega} K^*_{\omega}\, z_{\omega} / \sum_{\omega} |K_{\omega}|^2. \tag{12}$$

As in earlier treatments (e.g. W09), the frequency argument $[\omega]$ in the above expressions is confined to the grid defined by a discrete Fourier transform of duration T and to the common interval within which φ and φ' samples are selected.

The simple, ω-independent form of the covariance matrix of the scaled residual data and the above expression for the coupling parameter estimate imply that the real and imaginary parts of the coupling estimate are uncorrelated. For the case $\alpha'\omega' \neq \alpha\omega$, the variances are given by

$$\sigma^2[\mathrm{Re}(\hat{\lambda})] \approx \sigma^2[\mathrm{Im}(\hat{\lambda})] \approx (2I)^{-1}, \tag{13}$$

where

$$I = \sum_{\omega} |K_{\omega}|^2 \approx \int \frac{d\omega}{\Delta\omega} |K_{\omega}|^2, \tag{14}$$

$\Delta\omega = 2\pi/T$ being the (angular) frequency resolution of the observations. For the power-spectral case $\alpha'\omega' = \alpha\omega$, λ is real and

$$\sigma^2[\hat{\lambda}] \approx I^{-1}. \tag{15}$$

Large-scale flows couple only modes which differ in ℓ and m by small integers. For concreteness, I will assume that ℓ and m differ by no more than 4. Similarly, according to Equation (11), data sensitivity decreases with the frequency separation of the coupled modes, for slowly-varying flows. It is therefore appropriate to consider only couplings of modes of similar frequency. Focusing on modes of degree greater than about 30, which are of prime interest for the convection zone, the restriction to nearly-resonant couplings is equivalent to ignoring couplings between modes of different radial order. A further approximation to the data sensitivity expression can be obtained by exploiting the fact that the mode parameters vary smoothly with ℓ and m at fixed n. By Equation (2),

$$E[|a^{\alpha}_{\omega}|^2]_0 = |R^{\alpha}_{\omega}\, s^{\alpha}_{\omega}|^2, \tag{16}$$

where $(s^{\alpha}_{\omega})^2 \equiv E[|x^{\alpha}_{\omega}|^2]_0$ is the power spectrum of the source function. Equation (11) can therefore be rewritten

$$K_{\omega} \approx -2(\omega R + \omega' R'^*) R^* R' s^2 / \sqrt{P' P}, \tag{17}$$

where s' ($\equiv s'^{\alpha'}_{\omega'}$) has been set equal to s ($\equiv s^{\alpha}_{\omega}$) in accordance with assumed broad-band nature of the source spectrum. Using Equations (1) and (16) to replace P and P' in the above sensitivity equation, one obtains

$$|K_{\omega}|^2 \approx \frac{|2\omega(R + R'^*)|^2}{(1 + B|Rs|^{-2})(1 + B|R's|^{-2})}, \tag{18}$$

where the broad-band assumption has similarly been applied to the background spectra. In obtaining this and subsequent expressions, the near equality of the frequencies ω_α, $\omega_{\alpha'}$, ω, and ω' has been exploited.

By Equations (3) and (16) and the near equality of ω and ω_α and of $\gamma \equiv \gamma_\alpha$ and $\gamma_{\alpha'}$, the quantity $A \equiv (s/\gamma\omega)^2$ can be treated as a frequency-independent parameter equal to the peak power density of the mode-amplitude spectrum $E[|a^\alpha_\omega|^2]$. Therefore the amplitude spectra of the coupled modes can be written approximately as

$$|Rs|^2 = A\,|\gamma\omega R|^2 \tag{19}$$

and

$$|R's|^2 = A\,|\gamma\omega R'|^2. \tag{20}$$

In the same spirit, one obtains

$$2\omega R \approx -(\omega - \omega_0 + \mathrm{i}\frac{\gamma}{2})^{-1} \tag{21}$$

and

$$2\omega R' \approx -(\omega - \omega'_0 + \mathrm{i}\frac{\gamma}{2})^{-1} \tag{22}$$

from Equation (3), where $\omega_0 \equiv \omega_\alpha$ and $\omega'_0 \equiv \omega_{\alpha'} - \sigma$. The last pair of equations are conveniently rewritten as

$$\gamma\omega R \approx -(x + a + \mathrm{i})^{-1} \tag{23}$$

and

$$\gamma\omega R' \approx -(x - a + \mathrm{i})^{-1}, \tag{24}$$

where $\bar{\omega} \equiv (\omega_0 + \omega'_0)/2$, $x \equiv 2(\omega - \bar{\omega})/\gamma$, and $a \equiv (\omega'_0 - \omega_0)/\gamma$. The above equations imply that

$$|Rs|^{-2} = \frac{(x+a)^2 + 1}{A} \tag{25}$$

and

$$|R's|^{-2} = \frac{(x-a)^2 + 1}{A}. \tag{26}$$

Applying the previous pair of equations to the right side of Equation (18) gives

$$|K_\omega|^2 \approx \frac{(2/\gamma)^2\,(b^2-1)^2\,|f_1 + f_2|^2}{g_1\,g_2}, \tag{27}$$

where $b^2 \equiv 1 + A/B$, $f_1 = (x + a + \mathrm{i})^{-1}$, $f_2 = (x - a - \mathrm{i})^{-1}$, $g_1 = (x + a)^2 + b^2$, and $g_2 = (x-a)^2 + b^2$. Hereafter, the background spectrum [B] will be treated as a frequency-independent parameter, like A, and b is therefore a measure of the signal-to-noise of the mode-frequency measurement (Libbrecht, 1992). Substituting this last result into Equation (14) and recalling that $x \equiv 2(\omega - \bar{\omega})/\gamma$, one obtains

$$I \approx \frac{2(b^2-1)^2}{\gamma\Delta\omega} \int_{-\infty}^{\infty} \frac{|f_1 + f_2|^2}{g_1\, g_2}\, dx. \tag{28}$$

The extension of the limits of integration to plus and minus ∞ is justifiable on the grounds that the integral converges rapidly. Evaluation of the infinite integral yields the expression

$$I^{-1} \approx f(a,b)\, \frac{\gamma\Delta\omega}{4\pi}, \tag{29}$$

where

$$f(a,b) = \frac{b(1+b)[4a^2 + (1+b)^2]}{(b^2-1)^2}. \tag{30}$$

This expression, obtained using the "Mathematica" computer algebra package (Wolfram Research, Inc., 2012), agrees, for the case of vanishing background power (corresponding to the limit of infinite b), with an independent calculation based on contour integration. It is straightforward to verify that the expression for the variance of the frequency measurement, obtained from Equations (15), (29), and (30) with $a = 0$, is equivalent to that of Libbrecht (1992) for arbitrary b.

4. Inverting for Individual Spherical-Harmonic-Frequency Components of the Flow Velocity

Further insight is obtained by considering a spherical-harmonic representation of the flow velocity field. Following Lavely and Ritzwoller (1992, hereafter LR92), $\boldsymbol{u}_\omega = \boldsymbol{u}_\omega(r,\theta,\phi)$ can be decomposed into poloidal [$\mathbf{P}$] and toroidal [$\mathbf{T}$] harmonics, given by

$$\mathbf{P}^t_{s,\omega} = \hat{\mathbf{r}} u^t_{s,\omega}(r) Y^t_s(\theta,\phi) + v^t_{s,\omega}(r)[\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{\partial_\phi}{\sin\theta})] Y^t_s(\theta,\phi) \tag{31}$$

and

$$\mathbf{T}^t_{s,\omega} = -w^t_{s,\omega}(r)\hat{\mathbf{r}} \times [\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{\partial_\phi}{\sin\theta})] Y^t_s(\theta,\phi) \tag{32}$$

where r, θ, ϕ are spherical-polar coordinates, ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$) are the corresponding unit vectors, and Y^t_s are the scalar spherical harmonic functions of degree s and order t. Equation (C31) of LR92 can be rewritten in the form

$$\lambda^{n\ell' m'}_{n\ell m}(\sigma) = \sum_s b^{n\ell\ell'}_{st,\sigma}\, (-1)^{m'} \begin{pmatrix} \ell' & s & \ell \\ -m' & t & m \end{pmatrix} \sqrt{2s+1}, \tag{33}$$

where $\begin{pmatrix} \ell' & s & \ell \\ -m' & t & m \end{pmatrix}$ are Wigner $3-j$ symbols (Edmonds, 1960) and $t = m' - m$. According to Equations (C31-C34) of LR92, the measurable coefficients $b^{n\ell\ell'}_{st,\sigma}$ depend only on the radial profiles $u^t_{s,\sigma}(r)$, $v^t_{s,\sigma}(r)$, and $w^t_{s,\sigma}(r)$ of the flow, with a linear sensitivity of the form

$$b^{n\ell\ell'}_{st,\sigma} = \int (K_u\, u^t_{s,\sigma} + K_v\, v^t_{s,\sigma} + K_w\, w^t_{s,\sigma}) dr, \tag{34}$$

where the kernels $K_u(r)$, $K_v(r)$, $K_w(r)$ depend on n, ℓ, ℓ', and s, but not on t or σ.

For fixed n, ℓ, ℓ', t, and σ and with $m' = m + t$, Equation (33) can be written more concisely as

$$\lambda_m = \sum_s b_s\, \gamma^s_m, \tag{35}$$

where

$$\lambda_m \equiv \lambda^{n\ell'm'}_{n\ell m}(\sigma), \tag{36}$$

$$b_s \equiv b^{n\ell\ell'}_{st,\sigma}, \tag{37}$$

and

$$\gamma^s_m \equiv (-1)^{m'} \begin{pmatrix} \ell' & s & \ell \\ -m' & t & m \end{pmatrix} \sqrt{2s+1}. \tag{38}$$

Note that γ^s_m and λ_m are defined only for m such that $|m| \leq \ell$ and $|m+t| \leq \ell'$.

For each s, the γ^s_m are the components of a vector and, by the orthogonality property of the $3-j$ (Equation 3.7.8 of Edmonds, 1960), the vectors form an orthonormal set. Therefore the b-coefficients are just the projections of the vector $[\lambda_m]$ of couplings onto the γ^s_m:

$$b_s = \sum_m \gamma^s_m\, \lambda_m. \tag{39}$$

Aside from normalization, these equations generalize the familiar a-coefficient expansion of the m-dependence of the eigenfrequencies due to differential rotation (Schou *et al.*, 1998).

The analogous projection of the coupling *data* $[\hat{\lambda}_m]$ onto the orthonormal set yields estimates, $\hat{b}_s$, of the b-coefficients, which by Equation (34) are each sensitive to precisely one spherical-harmonic component of the flow velocity. The form of the covariance matrix of the projected coupling data follows straightforwardly from that of the coupling data. It can be argued that the covariance matrix of the $\hat{\lambda}_m$ is approximately proportional to the identity matrix, as follows. The frequencies ω_0 and ω'_0, in Equations (21) and (22), are respectively equal to $\omega_{n\ell m}$ and $\omega_{n\ell',m+t} + \sigma$ and $\gamma = \gamma_{n\ell m}$. Since the (approximately linear) dependence of $\omega_{n\ell m}$ on m, due mainly to solar differential rotation, varies slowly with ℓ, the frequency difference $\omega'_0 - \omega_0$ is very insensitive to m. Similarly ignoring the m-dependent contribution to the observed mode linewidth (due mainly to solar activity), one finds that the a-parameter of Equation (29), defined to be $(\omega_0' - \omega_0)/\gamma$, is approximately independent of m. The b-parameter of Equation (29) depends on the ratio of the peak power density [A] of the mode amplitude spectrum and the background power parameter [B]. The peak power parameter is observed to be approximately independent of m. In the interest of simplicity and because the signal-to-noise ratio is fairly high for the bulk of the observed solar oscillations, I ignore the m-dependence of the background power [B]. Thus in ideal data, the covariance matrix of the data [$\hat{\lambda}_m$], expressions for which were developed in the preceeding section, should indeed approximate the identity matrix times an m-independent factor. It then follows, because the $\hat{b}_s$ are $\hat{\lambda}_m$ projected onto an orthonormal set of vectors, that their covariance matrix is also the identity matrix times same the scale factor, namely

$$\sigma^2[\mathrm{Re}(\hat{b}_s)] \approx \sigma^2[\mathrm{Re}(\hat{\lambda}_m)], \tag{40}$$

where, by the preceeding argument, m is any value of the azimuthal order for which λ_m is defined.

The simplicity of the sensitivity function and the covariance matrix of the b-data implies that the task of inverting ideal data for a general flow can be separated into inversions for individual spherical-harmonic-frequency components. I now consider in some detail the problem of obtaining either $\mathbf{P}^t_{s,\sigma}$ or $\mathbf{T}^t_{s,\sigma}$ from observed b-coefficients. The problem of estimating the internal angular velocity has, of course, been studied in great detail (Thompson *et al.*, 2003; Howe, 2009). Of particular interest are the so-called 1.5-dimensional inversions, which use the conventional a-coefficient parameterization of the mode-frequency splittings to constrain steady, north-south-symmetric differential rotation (*e.g.* Schou *et al.*, 1998). Specifically, the $a_s(n,\ell)$ coefficients, which are proportional to $b^{n\ell\ell}_{s0,0}$, yield the profiles $w_s \equiv w^0_{s,0}$, for odd s. W12 indicated how "generalized" a-coefficients, which are basically the $b^{n\ell\ell'}_{s0,0}$ coefficients, also constrain the w_s profiles and, in addition, the $v_s \equiv v^0_{s,0}$ profiles of steady, axisymmetric meridional flow. By Equation (34), the data available at a particular n and ℓ to constrain a specific flow component, either $\mathbf{P}^t_{s,\sigma}$ or $\mathbf{T}^t_{s,\sigma}$, are $b^q \equiv b^{n\ell\ell'}_{st,\sigma}$, with $\ell' = \ell + q$. The b^q are defined only for q obeying the inequality $|q| \leq s$ and the conditions that $s + q$ must be even (odd) for poloidal (toroidal) components (Equations (118) and (121) of LR92). For the case $t = \sigma = 0$ of steady, axisymmetric flow, q is further restricted to being non-negative, as $\hat{b}^{-q}$ and $\hat{b}^q$, being related in this case by complex conjugation (as a consequence of their definition in terms of cross-covariance), are redundant data.

Simple dimensional analysis suggests that, for $s \leq 4$, poloidal flows deep in the convection zone satisfying the mass-flux constraint are dominated by horizontal motions (see also Chatterjee and Antia (2009)). Accordingly, the remainder of the article will ignore oscillations of degree greater than 4 and will focus on the problem of estimating only the horizontal flow velocity. (By ignoring the radial velocity component one of course underestimates the detectability of a flow.) For oscillation modes of $\ell >> s$, which are of prime interest in probing the convection zone, the sensitivity kernels K_v and K_w for a given n and ℓ (Equation (34)) are approximately proportional to one another. Useful, high-ℓ expressions for these kernels, which do not depend on t or σ, can be gleaned from treatments of the effects of steady axisymmetric flows on the mode eigenfunctions. From Appendix A of Vorontsov (2011), for instance, and the foregoing discussion of the b-coefficients, one obtains the asymptotic expressions

$$K_w(r) = f(s,q)\,K_{n\ell}(r) \tag{41}$$

and

$$K_v(r) = \mathrm{i}g(s,q)\,K_{n\ell}(r), \tag{42}$$

where

$$rK_{n\ell(r)} = \frac{(-1)^l}{\sqrt{2\pi}}\ell^{3/2}\,\rho(r)r^2[U_{n\ell}^2(r) + \ell(\ell+1)V_{n\ell}^2(r)], \tag{43}$$

with ρ denoting the solar mass density profile and $U_{n\ell}$ and $V_{n\ell}$ the radial and horizontal amplitudes of the oscillation eigenfunctions. For odd $s+q$,

$$f(s,q) = (-1)^{\frac{s+q-1}{2}}\,\frac{(s-q)!!\,(s+q)!!}{\sqrt{(s-q)!(s+q)!}}, \tag{44}$$

and $g(s,q)$ is zero while, for even $s+q$,

$$g(s,q) = (-1)^{\frac{s+q}{2}}\,q\,\frac{(s-q-1)!!\,(s+q-1)!!}{\sqrt{(s-q)!(s+q)!}} \tag{45}$$

and $f(s,q)$ is zero.

Therefore, as an approximation, the b^q can be regarded as independent measurements of $\int K_{n\ell}(r)\,v_{s,\sigma}^t(r)\,dr$, for even $s+q$, and of $\int K_{n\ell}(r)\,w_{s,\sigma}^t(r)\,dr$, for odd $s+q$, from which $v_{s,\sigma}^t(r)$ and $w_{s,\sigma}^t(r)$ can be obtained using standard one-dimensional inversion procedures. From Equations (31) and (32), and ignoring radial motion, follows that a single spherical-harmonic-frequency component of the flow contributes $|\tilde{v}_{s,\sigma}^t(r)|^2 + |\tilde{w}_{s,\sigma}^t(r)|^2$ to the mean-square velocity at radius r, where

$$(\tilde{v}_{s,\sigma}^t, \tilde{w}_{s,\sigma}^t) = \sqrt{\frac{s(s+1)}{2\pi}}(v_{s,\sigma}^t, w_{s,\sigma}^t) \tag{46}$$

unless $t = \sigma = 0$, in which case

$$(\tilde{v}_{s,\sigma}^t, \tilde{w}_{s,\sigma}^t) = \sqrt{\frac{s(s+1)}{4\pi}}(v_{s,\sigma}^t, w_{s,\sigma}^t). \tag{47}$$

The rescaled velocity profiles are truer measures of the flow velocity and the b^q-coefficients can be treated as constraints on $\int K_{n\ell}(r)\,\tilde{v}_{s,\sigma}^t(r)\,dr$ ($\int K_{n\ell}(r)\,\tilde{v}_{s,\sigma}^t(r)\,dr$), for even (odd) $s+q$.

The toroidal kernel K_w is a purely real function of r and the poloidal kernels K_u, K_v are purely imaginary. Consequently, the real or imaginary part of each velocity profile affects only one part, either real or imaginary, of the b^q. And, since the real and imaginary parts of these data are uncorrelated, the inversions for the real and imaginary velocity profiles can be performed independently. Letting η^q denote either the real or imaginary part of b^q and

$$\xi \equiv \int K_{n\ell}(r)\,\theta(r)\,dr, \tag{48}$$

where $\theta(r)$ is the part of $\tilde{v}^t_{s,\sigma}(r)$ or $\tilde{w}^t_{s,\sigma}(r)$ on which η^q depends, one finds that

$$\eta^q \approx G_q \xi, \tag{49}$$

where G_q is computable from Equations (34) and (41) through (48). A suitable datum for a one-dimensional inversion for $\theta(r)$ would thus be the conventional least-squares estimate of ξ, namely

$$\hat{\xi} = \sum_q \frac{G_q}{\sigma_q} \hat{y}_q / \sum_q \frac{G_q^2}{\sigma_q^2}, \tag{50}$$

with standard deviation

$$\sigma_\xi = [\sum_q (G_q/\sigma_q)^2]^{-1/2}, \tag{51}$$

where σ_q is the standard deviation of η^q (see discussion of Equation (40)), $\hat{y}_q \equiv \hat{\eta}_q/\sigma_q$, and sums over q are carried out over the allowed range as discussed above.

Since helioseismic inversions typically use data in which oscillations dominate the observed signal, it is illuminating to consider the problem of flow detectability in the approximation where the background component of the oscillation signal is ignored. Although the "no-background" approximation naturally leads to an underestimate of flow-measurement noise, it has the advantage of yielding the simple expression

$$\sigma_\xi^2 = \frac{\sigma_{n\ell}^2}{2\pi}, \text{ where } \sigma_{n\ell}^2 \equiv \frac{\gamma_{n\ell}\Delta\omega}{4\pi}, \tag{52}$$

for the variance in $\hat{\xi}$, where $\gamma_{n\ell}$ is the damping rate of modes of order n and degree ℓ and $\Delta\omega$ is the frequency resolution of the observations, as in Equation (29). The preceeding relation can be verified by direct calculation using Equation (51) and previous results.

In the no-background and high-ℓ approximations, the sensitivity and standard deviation of the ξ-data depend only on n and ℓ and are therefore the same for all flow components $[\theta(r)]$. This circumstance suggests that what has been learned about the statistics of the angular velocity measurement can be immediately applied to other components of the large-scale flow velocity. A number of studies suggest that the differential rotation can be measured with a standard deviation $\sigma_\theta \sim 1-2\ \mathrm{m\,s^{-1}}$ near the base of the convection zone (*e.g.* Schou *et al.*, 1998), and of course much more accurately near the surface, with 11 years of seismic data. Similar levels of measurement accuracy also appear to be attainable for axisymmetric meridional flow (*e.g.* Braun and Birch, 2008) and, by implication, for any of the profiles $\theta(r)$ of large-scale flow, with datasets of solar-cycle duration.

It may be worth noting that the detectability of general flow components can be increased beyond these estimates by demanding less depth resolution than is conventionally demanded for the angular velocity.

Finally, I consider the integrated large-scale flow-velocity power, which may be possible to detect even if individual velocity components are below the threshold of detectability. If $\hat{\theta}(r)$ is an unbiased estimate of $\theta(r)$, obtained for example from an inversion, then $\hat{\theta}^2(r)$ is a (biased) estimate of the contribution of $\theta(r)$ to the mean-square flow velocity at radius r. Suppressing the argument r, the expected value of $\hat{\theta}^2$ is $\theta^2 + \sigma_\theta^2$ and one might reasonably suppose σ_{θ^2}, the threshold for detecting power in one flow "channel", to be approximately $\sqrt{2}\sigma_\theta^2$, as would be the case if $\hat{\theta}$ were normally distributed. By combining the power estimates of N uncorrelated channels, the uncertainty in the mean power per channel becomes approximately $\sqrt{2/N}\,\sigma_\theta^2$. Thus the detection threshold for root-mean-square integrated velocity power is roughly $(2N)^{1/4}\,\sigma_\theta$. The number of independent flow channels should be the number of spherical-harmonic functions used to characterize the angular dependence of the flow times the number of frequency channels used to characterize the time dependence. Concentrating on the degree range $1 \le s \le 4$, for deep convection, yields 24 independent functions. The number of frequency channels should be roughly the duration of the observations divided by the characteristic time scale of the flow. With an 11-year database and a characteristic time of about one month (the estimated turnover time of the largest convective eddies), $N \sim 2500$ and large-scale flows with a root-mean-square integrated velocity power of a few tens of $\mathrm{m\,s^{-1}}$ may ultimately be detectable deep in the convection zone.

5. Discussion and Summary

A theoretical analysis of the noise in helioseismic waveform measurements of large-scale solar subsurface mass flows was presented in the preceeding sections. For conceptual ease, the waveform procedure on which the noise analysis was based proceeds by analogy with global-mode frequency-splitting analysis. In the first step, estimates of the flow-dependent couplings of global oscillation modes of the same radial order were obtained from covariance data by least-squares fitting. Included in the couplings are the frequency splittings themselves, obtained from the power-spectral subset of the data. Orthonormal combinations, referred to here as "b"-coefficients, of the measured mode couplings are then computed. In the case of frequency splittings, the projected couplings are proportional to the conventional a-coefficients. The projected coupling coefficients are each sensitive to precisely one spherical-harmonic-frequency component of either the toroidal or poloidal part of the flow velocity. As in the analysis of frequency-splitting measurements, this separability permits the $\mathbf{P}^t_{s,\sigma}$ and $\mathbf{T}^t_{s,\sigma}$ to be inverted for independently.

Section 2 summarizes the noise model of the raw covariance data. The model is based on the assumption that the uncertainty in the purely seismic (as opposed to background) component of the nominally seismic signal reflects mainly uncertainty in the source of the seismic waves. It should be noted that if additional, more-direct measurements of the wave source could be made, then the prospects of detecting deep flows would undoubtedly improve. In Section 3, the covariance matrix of the mode-coupling measurements was shown to be approximately diagonal and an analytical expression was derived for the measurement variance. As discussed earlier, the analysis underestimates the coupling measurement noise because it ignores error in the mode-parameter measurements. The projected coupling data were defined in Section 4 and their covariance matrix was shown to have an approximately diagonal form. It was then shown, using a high-ℓ approximation for data sensitivity, that the b-data can be compressed into a set of n- and ℓ-dependent ξ-data more analogous to the a-coefficients used in frequency-splitting analysis. This circumstance enables insights gained from the analysis of angular-velocity measurements to be applied to general large-scale flows.

To avoid undue complication, the noise analysis treated flows as small perturbations about a static solar model. (Note that helioseismic waveform analysis considered here assumes the Born approximation.) One might, however, be concerned about the adequacy of the present treatment of noise for modes of high-degree, for which the non-linearity in the eigenfunction dependence on the angular velocity cannot be neglected (*e.g.* Vorontsov, 2011; W12). To improve the convergence of the Born series one might instead perturb about a differentially-rotating reference model. In fact it can be shown that the present treatment is adequate, provided that the flow velocity, the eigenfunctions and their amplitudes, and related observables are interpreted in terms of a rotating model. In particular, the key expression (Equation (5)) for coupling-coefficient sensitivity retains its validity. Furthermore, little accuracy is lost if the eigenfunctions of the static reference model are substituted in the sensitivity expression for those of the rotating model. The justification for this replacement is that, for typical (ℓ, m) values, the latitude dependence of the integrand in Equation (5) is the sum of a part which oscillates rapidly about zero, whose form reflects the precise latitudinal nodal structure of the eigenfunctions, and a relatively smooth envelope. At sufficiently high ℓ, the latitudinal remapping of nodal structure implied by differential rotation can drastically alter the oscillating part. The integral, however, is dominated by the envelope.

There are a number of ways in which the treatment of noise could be improved. From the standpoint of analyzing actual data, the biggest shortcoming of the analysis is that it ignores leakage effects due to the grossly uneven photospheric sampling of contemporary datasets. Spatial leakage greatly complicates the flow measurement problem, particularly because it spoils the separability of the inversion problem, on which the present analysis is based. Nevertheless, the flow measurement problem can at least be addressed with standard linear-inversion techniques. Other assumptions and approximations made in the present noise analysis, e.g. the neglect of uncertainty in mode parameters, can be relaxed within the framework of linear inverse theory and in fact the framework appears to be sufficiently general for dealing with not-so-large-scale flows. It would also be of interest to investigate the basic assumption about the Gaussianity of the wave field. Such studies might best be carried out using numerical simulations of wave propagation.

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